PART A TOPOLOGY COURSE: HT 2018

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Contents

Recommended books in Topology:

(1) M.A. Armstrong, *Basic Topology*;
(2) J.R. Munkres, *Topology, A First Course*;
(3) W. Sutherland, *Introduction to Metric and Topological Spaces* (particularly recommended for the first half of the course);

For further reading see the list at the end of the synopsis on the Maths Institute web pages.

1. Topological spaces

1.1. **What is Topology about?** You have already studied metric spaces in some detail. These are objects where one has a notion of distance between points, satisfying some simple axioms. They have a rich and interesting theory, which leads to such concepts as connectedness, completeness and compactness.

Two metric spaces are viewed as 'the same' if there is an isometry between them, which is a bijection that preserves distances. But there is a much more flexible notion of equivalence: two spaces are homeomorphic if there is a continuous bijection between them with continuous inverse. Many properties of metric spaces are preserved by a homeomorphism (for example, connectedness and compactness). Thus homeomorphic metric spaces may have very different metrics, but nevertheless have many properties in common. The conclusion to draw from this is that a metric is, frequently, a somewhat artificial and rigid piece of structure. So, one is led naturally to the study of Topology. The fundamental objects in Topology are topological spaces. Here, there is no metric in general. But one still has a notion of open sets, and so concepts such as connectedness and compactness continue to make sense.

Why study Topology? The reason is that it simultaneously simplifies and generalises the theory of metric spaces. By discarding the metric, and focusing solely on the more basic and fundamental notion of an open set, many arguments and proofs are simplified. And many constructions (such as the important concept of a quotient space) cannot be carried out in the setting of metric spaces: they need the more general framework of topological spaces. But perhaps the most important reason is that the spaces that arise naturally in Topology have a particularly beautiful theory. Nowhere is this more evident than in the study of surfaces, which are the focus of the final chapter of this course.

1.2. **Definitions and examples.**

Definition 1.1. A *topological space* $(X, \mathcal{T})$ consists of a non-empty set X together with a family $\mathcal{T}$ of subsets of X satisfying:

(T1) $X, \emptyset \in \mathcal{T}$;

(T2) $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$;

(T3) $U_i \in \mathcal{T}$ for all $i \in I$ (where I is some indexing set) $\Rightarrow \bigcup_{i \in I} U_i \in \mathcal{T}$.

The family $\mathcal{T}$ is called a *topology* for X. The sets in $\mathcal{T}$ are called *the open sets of* X. When $\mathcal{T}$ is understood we talk about *the topological space* X.

Remarks 1.2. A consequence of (T2) is that if $U_1, \dots, U_n$ is a collection of open sets, then $U_1 \cap \cdots \cap U_n$ is open. But the intersection of infinitely many open sets need not be open!

On the other hand, in (T3), the indexing set I is allowed to be infinite. It may even be uncountable.

In some sense, this is a confusing definition. When studying a topological space, we simply *declare* at the outset what its open sets are. One might ask *why* is a set open? The answer is that we have simply declared it to be so.

Nevertheless, there is an important situation where the open sets arise naturally: in a metric space. Let us recall some definitions from the course on Metric Spaces:

Definition 1.3. A *metric space* is a non-empty set X endowed with a function $\mathrm{d} : X \times X \to \mathbb{R}$ with the following properties:

(M1) $\mathrm{d}(x, y) \geqslant 0$ for all $x, y \in X$; $\mathrm{d}(x, y) = 0$ if and only if $x = y$;

(M2) (Symmetry) for all $x, y \in X$, $\mathrm{d}(y, x) = \mathrm{d}(x, y)$;

(M3) (Triangle inequality) for all $x, y, z \in X$, $\mathrm{d}(x, z) \leqslant \mathrm{d}(x, y) + \mathrm{d}(y, z)$.

The function d is called a *metric* or *distance*.

Definition 1.4. Let (X, d) be a metric space, let x be a point in X and let $\epsilon > 0$. Then the *ball of radius ϵ about x*, denoted $B(x,\epsilon)$, is $\{y \in X : \mathrm{d}(x,y) < \epsilon\}$.

Definition 1.5. Let (X, d) be a metric space and $U \subseteq X$. We say U *is open in X* if for every $x \in U$, there exists $\varepsilon_x > 0$ such that $B(x, \varepsilon_x) \subseteq U$.

Proposition 1.6. *Let (X, d) be a metric space. Then the open subsets of X form a topology, denoted $\mathcal{T}_{\mathrm{d}}$.*

Proof. (T1) is trivial.

(T2) Let U and V be open subsets of X. Consider an arbitrary point $x \in U \cap V$.

As U is open there exists $\alpha_x > 0$ such that $B(x, \alpha_x) \subseteq U$. Likewise, as $x \in V$ and V is open there exists $\beta_x > 0$ such that $B(x, \beta_x) \subseteq V$.

Then for $\delta_x = \min(\alpha_x \,, \beta_x)$, $B(x, \delta_x) \subseteq B(x, \alpha_x) \subseteq U$ and $B(x, \delta_x) \subseteq B(x, \beta_x) \subseteq V$. Therefore $B(x, \delta_x) \subseteq U \cap V$.

(T3) For every $x \in \bigcup_{i \in I} U_i$ there exists $k \in I$ such that $x \in U_k$. Since U_k is open there exists $\varepsilon_x > 0$ such that $B(x, \varepsilon_x) \subseteq U_k \subseteq \bigcup_{i \in I} U_i$. $\square$

The following are some other examples of topological spaces.

Examples 1.7.
1. (Discrete spaces) Let X be any non-empty set. The *discrete topology* on X is the set of *all* subsets of X.
2. (Indiscrete spaces) Let X be any non-empty set. The *indiscrete topology* on X is the family of subsets $\{X, \emptyset\}$.
3. Let X be any non-empty set. The *co-finite* topology on X consists of the empty set together with every subset U of X such that $X \setminus U$ is finite.

Definition 1.8.
1. We call a topological space $(X, \mathcal{T})$ *metrizable* if it arises as in Proposition 1.6, from (at least one) metric space (X, d) i.e. there is at least one metric d on X such that $\mathcal{T} = \mathcal{T}_{\mathrm{d}}$.
2. Two metrics on a set are *topologically equivalent* if they give rise to the same topology.

Examples 1.9.
1. The metrics $\mathrm{d}_1, \mathrm{d}_2, \mathrm{d}_\infty$ on $\mathbb{R}^n$ are all topologically equivalent. (Recall that $\mathrm{d}_1, \mathrm{d}_2, \mathrm{d}_\infty$ are the metrics arising from the norms $\| \, \|_1, \| \, \|_2, \| \, \|_\infty$, respectively.)

   We shall call the topology defined by the above three metrics the *standard* (or *canonical*) topology on $\mathbb{R}^n$.
2. The discrete topology on a non-empty set X is metrizable, using the metric
$$\mathrm{d}(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$
   It is easy to check that this is a metric. To see that it gives the discrete topology, consider any subset $U \subseteq X$. Then for every $x \in U$, $B(x, \frac{1}{2}) \subseteq U$.

Definition 1.10. Given two topologies $\mathcal{T}_1, \mathcal{T}_2$ on the same set, we say $\mathcal{T}_1$ is *coarser than* $\mathcal{T}_2$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

Remark 1.11. For any space $(X, \mathcal{T})$, the indiscrete topology on X is coarser than $\mathcal{T}$ which in turn is coarser than the discrete topology on X.

Definition 1.12. Let $(X, \mathcal{T})$ be a topological space.

A subset V of X is *closed in* X if $X \setminus V$ is open in X (i.e. $X \setminus V \in \mathcal{T}$).

Examples 1.13. (a) You have to be careful which space you're saying a set is closed *in*.

For example in the space $[0, 1)$ with the usual topology coming from the Euclidean metric, $[1/2, 1)$ is closed.

(b) In a discrete space, *all* subsets are closed since their complements are open.

(c) In the co-finite topology on a set X, a subset is closed if and only if it is finite or all of X.

Proposition 1.14. *Let X be a topological space. Then*

(C1) *$X, \emptyset$ are closed in X;*

(C2) *if V_1, V_2 are closed in X then $V_1 \cup V_2$ is closed in X;*

(C3) *if V_i is closed in X for all $i \in I$ then $\bigcap_{i \in I} V_i$ is closed in X.*

Proof. Properties (C1),(C2), (C3) follow from (T1), (T2), (T3) and from the De Morgan laws (see the Appendix). □

Definition 1.15. A sequence (x_n) in a topological space X *converges to a point* $x \in X$ if given any open set U containing x there exists an integer N such that $x_n \in U$ for all $n \geqslant N$.

Examples 1.16. (a) In a metric space this is equivalent to the metric definition of convergence. This was proved in the Metric Spaces course.

(b) In an indiscrete topological space X any sequence converges to any point $x \in X$.

(c) In an infinite space X with the co-finite topology any sequence (x_n) of pairwise distinct elements (i.e. such that $x_n \neq x_m$ when $n \neq m$) converges to any point $x \in X$.

Remark 1.17. Note that the uniqueness of the limit for a convergent sequence *is not granted in a topological space.*

In particular (b) and (c) show that both the indiscrete topology and the co-finite topology on an infinite set *are not metrizable*, as in a metric space any convergent sequence has only one limit.

In order to have the uniqueness of the limit for a convergent sequence, one has to *add an extra axiom (the Hausdorff condition).* We shall discuss this axiom later on.

Proposition 1.18. *If a subset F in a topological space X is closed then for any convergent sequence contained in F, any limit of it is also in F.*

Proof. Let (x_n) be a convergent sequence, contained in F, and let $x \in X$ be a limit of (x_n). If $x \in X \setminus F$ then, as $X \setminus F$ is open, it follows that there exists N such that $x_n \in X \setminus F$ for every $n \geq N$. This contradicts the hypothesis that (x_n) is contained in F. □

Remark 1.19. Unlike in the case of metric spaces, the converse of the statement in Proposition 1.18 *might not be true.*

Loose Remark. We shall see many implications of the following form:

TOPOLOGICAL PROPERTY $\Rightarrow$ PROPERTY IN TERMS OF SEQUENCES.

In metric spaces the converse implication may sometime be true (i.e. *the topological property may be characterized in terms of sequences*).

In the general topological setting *the converse implication is never true.*

Definition 1.20. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and that $f : X \to Y$ is a map. We say that f is *continuous* if $U \in \mathcal{T}_Y \Rightarrow f^{-1}(U) \in \mathcal{T}_X$ i.e. 'inverse images of open sets are open'.

If there are other topologies around we may say 'f is continuous with respect to $\mathcal{T}_X$, $\mathcal{T}_Y$' or 'f is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous'.

Again, when X and Y are metric spaces, the continuity of a function $f\colon X \to Y$ is equivalent to the usual definition using ϵ and δ. This was proved in the Metric Spaces course.

Proposition 1.21. *A map $f : X \to Y$ between topological spaces is continuous if and only if $f^{-1}(V)$ is closed in X whenever V is closed in Y.*

Proof. This follows from the definition of a continuous map and from the formula

$$f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)\,.$$

□

Proposition 1.22. *If $f : X \to Y$ and $g : Y \to Z$ are continuous maps, where X, Y, Z are topological spaces, then so is $g \circ f$.*

Proof. Let U be an open set in Z.

Since g is continuous, $g^{-1}(U)$ is an open subset of Y.

Since f is continuous, $f^{-1}\left(g^{-1}(U)\right)$ is an open subset of X.

But the latter is the same as $(g \circ f)^{-1}(U)$. □

Definition 1.23. A *homeomorphism* between topological spaces X and Y is a bijection $f : X \to Y$ such that f and f^{-1} are continuous. The spaces X and Y are then said to be *homeomorphic.*

Proposition 1.24. *If $f : X \to Y$ is a continuous map between two topological spaces, then, for any sequence (x_n) in X converging to a point x, $(f(x_n))$ converges to $f(x)$.*

Proof. Let U be an open set containing $f(x)$. Then $f^{-1}(U)$ is an open set containing x, because f is continuous.

Since (x_n) converges to x there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in f^{-1}(U)$. It follows that $f(x_n) \in U$. □

Remark 1.25. The converse of Proposition 1.24 might not be true.

The following simple result is often a really useful way of showing that a set is open.

Lemma 1.26. *Let U be a subset of a topological space. Then the following are equivalent:*

(1) *U is open;*
(2) *for every x in U, there is an open set U_x containing x such that that $U_x \subseteq U$.*

Proof. (1) $\Rightarrow$ (2): Suppose that U is open. For each $x \in U$, set $U_x = U$.

(2) $\Rightarrow$ (1): Suppose that for each $x \in U$, there is an open set U_x as in (2). Then clearly $\bigcup_{x \in U} U_x = U$. So, U is a union of open sets, and hence is open. □

1.3. **Closure of a set.** It is, of course, possible for a subset of a topological space to be neither open nor closed. But we will now define an associated subset that is necessarily closed.

Definition 1.27. Consider a topological space X and a subset $A \subseteq X$.

The *closure of* A *in* X is the set

$$\overline{A} = \bigcap_{F \text{closed}, A \subseteq F} F .$$

Proposition 1.28. *The closure* $\overline{A}$ *is* ***the smallest (with respect to inclusion) closed subset of*** X ***containing*** A*, that is:*

(a) $\overline{A}$ *is closed in* X *and it contains* A*;*

(b) *if* $A \subseteq V$ *where* V *is closed in* X *then* $\overline{A} \subseteq V$.

Proof. (a) is immediate from the definition of the closure and property (C3) of closed sets.

A closed subset V as in (b) appears in the intersection defining $\overline{A}$, hence $\overline{A} \subseteq V$. □

Proposition 1.29. *Let* A, B *be subsets of a topological space* X. *Then*

(a) $A \subseteq B$ *implies that* $\overline{A} \subseteq \overline{B}$*;*

(b) A *is closed in* X *if and only if* $\overline{A} = A$*;*

(c) $\overline{\overline{A}} = \overline{A}$.

Proof. (a) We have $A \subseteq B \subseteq \overline{B}$, and then we apply Proposition 1.28, (b).

(b) $A \subseteq \overline{A}$. If A is closed then Proposition 1.28, (b), with $V = A$ implies $\overline{A} \subseteq A$, whence equality.

If $A = \overline{A}$ then A is closed.

(c) We apply (b) to $\overline{A}$. □

The following is a way of characterising points in $\overline{A}$ that is often useful.

Proposition 1.30.

$$\overline{A} = \{x \in X \ : \text{ for every open } U \subseteq X \quad \text{with } x \in U\,,\, U \cap A \neq \emptyset\}.$$

Proof. Consider $A_1 = \{x \in X : \text{ for every open } U \subseteq X \quad \text{with } x \in U\,,\, U \cap A \neq \emptyset\}$.

We first prove that $\overline{A} \subseteq A_1$ using Proposition 1.28, (b). Clearly A_1 contains A. It remains to prove that A_1 is closed. We prove that $X \setminus A_1 = B_1$ is open, by using Lemma 1.26. Note that

$$B_1 = \{x \in X : \text{ there exists an open set } U_x \text{ such that } x \in U_x \subseteq X \setminus A\}\,.$$

For every element y in some U_x one can take $U_y = U_x$ and have $y \in U_y \subseteq X \setminus A$. It follows that each U_x is entirely contained in B_1. So, by Lemma 1.26, B_1 is open.

We have thus proved that A_1 is closed, which implies that $\overline{A} \subseteq A_1$.

Assume that there exists $x \in A_1 \setminus \overline{A}$. Since $x \notin \overline{A}$ it follows that there exists a closed set F containing A such that $x \notin F$. Then $x \in X \setminus F$, and $X \setminus F$ is an open set which does not intersect A. This contradicts the fact that $x \in A_1$. □

Examples 1.31. (1) The closure of (a, b) in $(\mathbb{R}, |\,|)$ is $[a, b]$.

(2) The closure of $\mathbb{Q}$ in $(\mathbb{R}, |\,|)$ is $\mathbb{R}$. Same for $\mathbb{R} \setminus \mathbb{Q}$.

(3) The closure of any set A in a space X endowed with the co-finite topology is either X if A is infinite, or A if A is finite.

Definition 1.32. A subset A in X is called *dense* if $\overline{A} = X$.

We shall see in what follows that the closure of a set A contains two types of points:

- points **in** A that are **isolated**;
- points **in** A **or outside** A but near which points in A **accumulate** (i.e. **accumulation points**).

Definition 1.33. A point $x \in X$ such that for any open $U \subseteq X$ with $x \in U$, $(U \setminus \{x\}) \cap A \neq \emptyset$ is called an *accumulation point (or limit point) of* A.

The set of accumulation points of A is sometimes denoted by A'.

Note that $\overline{A} = A \cup A'$. The points in $A \setminus A'$ are points $x \in A$ such that for some open set U containing x, $U \cap A = \{x\}$. Such points are called *isolated points of* A.

Examples 1.34. In the metric space $(\mathbb{R}, |\ |)$

(1) if $A = (a,b)$ then $A' = \overline{A} = [a,b]$;

(2) if $A = \mathbb{Q}$ or $\mathbb{R} \setminus \mathbb{Q}$ then $A' = \overline{A} = \mathbb{R}$;

(3) if $A = \mathbb{Z}$ then $\overline{A} = \mathbb{Z}$ while $A' = \emptyset$.

(4) if $A = (0,1) \cup \{9,10\}$ then $\overline{A} = [0,1] \cup \{9,10\}$, $A' = [0,1]$ and $9, 10$ are isolated points.

Proposition 1.35. *Let* (X, d) *be a metric space and let* $A \subseteq X$.

(1) *The closure* $\overline{A}$ *is* the set of limits of convergent sequences (a_n) in A.

(2) *The set of accumulation points* A' *is* the set of limits of convergent sequences (a_n) of pairwise distinct elements in A (i.e. such that $a_n \neq a_m$ if $n \neq m$).

Proof. (1) Let A_ℓ be the set of all limits of convergent sequences (a_n) in A.

Since $\overline{A}$ is closed and it contains A, it contains all the limits of convergent sequences (a_n) in A, due to Proposition 1.18. Thus $A_\ell \subseteq \overline{A}$.

On the other hand, let x be a point in $\overline{A}$. According to Proposition 1.30, for every integer $n \geq 1$, the open ball $B\left(x, \frac{1}{n}\right)$ intersects A in a point a_n. The sequence (a_n) converges to x. Hence $x \in A_\ell$. We have thus proved that $\overline{A} \subseteq A_\ell$.

(2) Let A'_ℓ be the set of all limits of convergent sequences (a_n) in A such that $a_n \neq a_m$ whenever $n \neq m$.

Let x be a point in A'_ℓ, the limit of a sequence (a_n). At most one element in the sequence can be equal to x, thus without loss of generality we may assume that $a_n \neq x$ for all n.

For every open set U containing x there exists N such that $a_n \in U$ for all $n \geq N$. Moreover $a_n \in U \setminus \{x\}$. In particular $(U \setminus \{x\}) \cap A \neq \emptyset$. Thus $x \in A'$.

We conclude that $A'_\ell \subseteq A'$.

Conversely let x be a point in A'. We construct inductively a sequence (a_n) in A such that $\mathrm{d}(x, a_{n+1}) < \mathrm{d}(x, a_n)$ and $0 < \mathrm{d}(x, a_n) < \frac{1}{n}$.

Since $x \in A'$, $B(x,1) \setminus \{x\}$ intersects A in a point a_1.
Assume that we found the elements in the sequence $a_1, ..., a_n$ satisfying the required hypotheses. Let $r_{n+1} = \min\left(\frac{1}{n+1}\, , \, \mathrm{d}(x, a_n)\right)$.

Since $x \in A'$, $B(x, r_{n+1}) \setminus \{x\}$ intersects A in a point a_{n+1}.

The sequence (a_n) thus constructed converges to x, and for $n < m$, $\mathrm{d}(x, a_m) < \mathrm{d}(x, a_n)$, whence $a_m \neq a_n$. Therefore $x \in A'_\ell$.

We have proved that $A' \subseteq A'_\ell$, which together with the converse inclusion implies that $A' = A'_\ell$. □

Proposition 1.36. *A map $f : X \to Y$ of topological spaces is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for every $A \subseteq X$.*

Proof. Assume that f is continuous. Since $\overline{f(A)}$ is closed, it follows that $f^{-1}\left(\overline{f(A)}\right)$ is closed. The latter subset also contains A, therefore it contains $\overline{A}$. We have thus proved that $\overline{A} \subseteq f^{-1}\left(\overline{f(A)}\right)$, which implies that $f(\overline{A}) \subseteq \overline{f(A)}$.

Conversely, assume that $f(\overline{A}) \subseteq \overline{f(A)}$ for every $A \subseteq X$.

Let F be a closed subset in Y and let $A = f^{-1}(F)$. Then $f(A) \subseteq F$, whence $f(\overline{A}) \subseteq \overline{f(A)} \subseteq F$. It follows that $\overline{A} \subseteq f^{-1}(F) = A$, therefore $\overline{A} = A$. This implies that A is closed. □

1.4. **Interior of a set.** Just as one can define a closed set $\overline{A}$ associated with any subset A of a topological space, one can also define an associated set that is guaranteed to be open.

Definition 1.37. Let X be a topological space X and A a subset in X. The *interior of A in X* is the set

$$\mathring{A} = \bigcup_{U \text{open}, U \subseteq A} U\,.$$

Note that $\mathring{A}$ may be empty even if A is not. For instance consider the subset $A = [0,1] \times \{0\}$ in $\mathbb{R}^2$. Clearly A contains no open ball $B((x,y),\varepsilon)$, therefore A contains no non-empty open subset.

Proposition 1.38. *The interior $\mathring{A}$ is* ***the largest (with respect to inclusion) open subset of X contained in*** *A, that is:*

(a) *$\mathring{A}$ is open and contained in A;*

(b) *every open subset of X contained in A lies in $\mathring{A}$.*

Both (a) and (b) follow easily from the definition of the interior.

Proposition 1.39. *Let A, B be subsets of a topological space X. Then*

(a) *$A \subseteq B$ implies that $\mathring{A} \subseteq \mathring{B}$;*

(b) *A is open in X if and only if $\mathring{A} = A$;*

(c) *$\mathring{\mathring{A}} = \mathring{A}$;*

Proof. (a) $\mathring{A} \subseteq A \subseteq B$, and $\mathring{A}$ is open. Therefore $\mathring{A} \subseteq \mathring{B}$.

(b) If A is open, since $A \subseteq A$ it follows that $A \subseteq \mathring{A}$, whence equality. The converse implication is immediate.

(c) follows from (b) applied to $\mathring{A}$.

□

Examples 1.40.

- The interior of $[a,b]$ (or $(a,b]$) in $(\mathbb{R},|\ |)$ is (a,b);

  Indeed (a,b) is open and contained in $[a,b]$, therefore in its interior.

  If a was in the interior then there would exist $\varepsilon > 0$ such that $(a-\varepsilon, a+\varepsilon)$ is in the interior, therefore in $[a,b]$. This is impossible, hence a is not in the interior. Likewise for b. It follows that the interior is in (a,b), therefore it is equal to it.

- The interiors of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}\setminus\mathbb{Q}$ in $(\mathbb{R},|\ |)$ are $\emptyset$.

  None of the sets above contains an open interval, therefore none contains a non-empty open subset.

- Let X be a space endowed with the co-finite topology, and let $A\subseteq X$. The interior $\mathring{A}$ is either equal to A if $X\setminus A$ is finite (i.e. A is open), or it is empty.

  Indeed if $\mathring{A}$ would be non-empty, then $X\setminus\mathring{A}$ would be finite.

  As $\mathring{A}$ is contained in A, $X\setminus A$ is contained in $X\setminus\mathring{A}$, therefore it is also finite.

Next we note that the operation of taking the interior and of taking the closure are in some sense *complementary to each other.*

Proposition 1.41. *Let A be a subset in a topological space X.*

The closure of the complement of A is the complement of the interior*, i.e.*

$$\overline{X\setminus A} = X\setminus\mathring{A}.$$

Proof.

$$\begin{aligned} X\setminus\mathring{A} &= X\setminus\left(\bigcup_{U \text{open}, U\subseteq A} U\right) \\ &= \bigcap_{U\text{open}, U\subseteq A} (X\setminus U) \\ &= \bigcap_{F\text{closed}, F\supseteq X\setminus A} F \\ &= \overline{X\setminus A}. \end{aligned}$$

□

If in Proposition 1.41 we denote $X\setminus A$ by B we obtain:

Proposition 1.42. *Let B be a subset in a topological space X.*

The interior of the complement of B is the complement of the closure*, i.e.*

$$\overset{\circ}{\widehat{X\setminus B}} = X\setminus\overline{B}.$$

Note that the notation $\overset{\circ}{\widehat{X\setminus B}}$ simply means the interior of $X\setminus B$.

1.5. **Boundary of a set.**

Definition 1.43. The set $\partial A = \overline{A} \setminus \mathring{A}$ is called the *boundary* of A.

Proposition 1.44. *The boundary of A equals the intersection $\overline{A} \cap \overline{X \setminus A}$, and it equals the boundary of $X \setminus A$.*

Proof. Indeed $\partial A = \overline{A} \setminus \mathring{A} = \overline{A} \cap (X \setminus \mathring{A}) = \overline{A} \cap \overline{(X \setminus A)}$. The last inequality is due to Proposition 1.41.

Then by the above $\partial(X \setminus A) = \overline{(X \setminus A)} \cap \overline{A}$.

We conclude that $\partial A = \partial(X \setminus A)$. □

Example 1.45. Let $A = [a, b)$, where $a < b$. Then $\overline{A} = [a, b]$.

On the other hand $\mathbb{R} \setminus A = (-\infty, a) \cup [b, +\infty)$, and its closure is $\overline{\mathbb{R} \setminus A} = (-\infty, a] \cup [b, +\infty)$.

It follows that $\partial A = \{a, b\}$.

1.6. **Separation axioms.**

Definition 1.46. A topological space *satisfies the first separation axiom* if for any two distinct points $a \neq b$ there exists an open set U containing a and not b.

Proposition 1.47. *A topological space satisfies the first separation axiom if and only if singletons are closed subsets in it.*

Proof. If every singleton $\{x\}$ is closed in X then for any two distinct points $a \neq b$ consider $U = X \setminus \{b\}$.

Conversely, assume that X satisfies the first separation axiom. We prove that $V = X \setminus \{x\}$ is open, for any $x \in X$.

Let $y \in V$ arbitrary. Then $y \neq x$, hence by the first separation axiom there exists U_y open, containing y, not containing x, hence contained in V.

So, by Lemma 1.26, V is open. □

Definition 1.48. A topological space X is said to be *Hausdorff* (or to *satisfy the second separation axiom*) if given any two distinct points x, y in X, there exist disjoint open sets U, V with $x \in U$, $y \in V$.

Examples 1.49.
(a) Any metric space is Hausdorff.
(b) An indiscrete space X with more than one point does not satisfy the first separation axiom.
(c) An infinite set with the co-finite topology is not Hausdorff (*consequently it is not a metrizable topological space*), but it satisfies the first separation axiom because singletons are closed in it.

Proposition 1.50. *If $f : X \to Y$ is an injective continuous map between topological spaces, and Y satisfies the first (respectively, second) separation axiom then so does X.*

Proof. Assume that Y satisfies the first separation axiom. Let a, b be two distinct points in X. Since f is injective, $f(a) \neq f(b)$. By the first separation axiom in Y there exists U open such that $f(a) \in U$ and $f(b) \notin U$. This implies that $a \in f^{-1}(U)$, and $b \notin f^{-1}(U)$. As f is continuous, $f^{-1}(U)$ is open. We have thus proved that X satisfies the first separation axiom.

A similar argument shows that if Y is Hausdorff then X is Hausdorff. Note that we must use the fact that $U \cap V = \emptyset \Rightarrow f^{-1}(U) \cap f^{-1}(V) = \emptyset$. See the Appendix. □

Corollary 1.51. *If spaces X, Y are homeomorphic then X satisfies the first (second) separation axiom if and only if Y is satisfies the first (respectively the second) separation axiom.*

Proposition 1.52. *In a Hausdorff space, a sequence converges to at most one point.*

Proof. Indeed assume that a convergent sequence (x_n) has two distinct limits $a \neq b$. Then by the Hausdorff property there exist U, V open and disjoint (i.e. $U \cap V = \emptyset$) such that $a \in U$ and $b \in U$. As (x_n) converges to a there exists N such that $x_n \in U$ for every $n \geq N$. Similarly, (x_n) convergent to b implies that there exists N' such that $x_n \in V$ for every $n \geq N'$. Then for every $n \geq \max(N, N')$, $x_n \in U \cap V$, contradicting $U \cap V = \emptyset$.

□

Remark 1.53. Note that the first separation axiom does not suffice to ensure the uniqueness of the limit of a convergent sequence. Indeed, an infinite space with the co-finite topology satisfies the first separation axiom (see above), but contains sequences with several limits (see Example 1.16, (c)).

1.7. **Subspace of a topological space.** In the Metric Spaces course, it was shown that a non-empty subset of a metric space is naturally endowed with a metric structure.

Now we shall see that the same thing happens for topological structures.

Definition 1.54. Let $(X, \mathcal{T})$ be a topological space and let A be a non-empty subset of X.

The *subspace* or *induced* topology on A is $\mathcal{T}_A = \{A \cap U : U \in \mathcal{T}\}$.

The three properties (T1), (T2), (T3) that a topology must satisfy are easily checked:

(T1) $\emptyset = \emptyset \cap A \in \mathcal{T}_A$; $A = X \cap A \in \mathcal{T}_A$.

(T2) Given V_1, V_2 in $\mathcal{T}_A$, $V_1 = A \cap U_1$ and $V_2 = A \cap U_2$, where U_1 and U_2 are both in $\mathcal{T}$.

Then $V_1 \cap V_2 = U_1 \cap U_2 \cap A \in \mathcal{T}_A$.

(T3) Let $(V_i)_{i \in I}$ be such that $V_i \in \mathcal{T}_A$ for every $i \in I$. Then for every $i \in I$ there exists U_i open subset in X such that $V_i = A \cap U_i$.

It follows that $\bigcup_{i \in I} V_i = A \cap \bigcup_{i \in I} U_i$ which is in $\mathcal{T}_A$ because $\bigcup_{i \in I} U_i$ is in $\mathcal{T}$.

The following is easy but useful.

Lemma 1.55. *Let $(X, \mathcal{T})$ be a topological space, and let A be a non-empty subset of X. Give A the subspace topology. Then the inclusion map $i \colon A \to X$ is continuous.*

Proof. Let U be an open subset of X. Then $i^{-1}(U) = A \cap U$, which is open in A. So, i is continuous. □

Proposition 1.56. *A subset W in A is closed in $\mathcal{T}_A$ if and only if there exists F closed in $(X, \mathcal{T})$ such that $W = A \cap F$.*

Proof. Assume that $W \subseteq A$ is closed in $\mathcal{T}_A$. Then $A \setminus W$ is open, i.e. there exists U open in X such that $A \setminus W = A \cap U$.

The last equality may be re-written as $W = A \setminus A \cap U = A \setminus U = A \cap (X \setminus U)$. The set $F = X \setminus U$ is closed in X.

Conversely, let F be closed in $(X, \mathcal{T})$, and let $W = A \cap F$.

Then $A \setminus W = A \setminus F = A \cap (X \setminus F)$. Since F is closed in X, $X \setminus F$ is open in X, hence $A \setminus W$ is open in A. It follows that W is closed in A. □

Proposition 1.57. *Let $(A, \mathcal{T}_A)$ be a subspace of $(X, \mathcal{T})$, and let $B \subseteq A$.*

(a) *The closure of B in $(A, \mathcal{T}_A)$, $\overline{B}^A$,* coincides *with the intersection between A and the closure of B in X, $\overline{B}^X$.*

(b) *The interior of B in $(A, \mathcal{T}_A)$, $\mathring{B}^A$,* contains *the interior of B in X, $\mathring{B}^X$.*

Proof. (a) The set $\overline{B}^X$ is closed in X, therefore $\overline{B}^X \cap A$ is closed in A, and it contains B. Therefore $\overline{B}^A \subseteq \overline{B}^X \cap A$.

On the other hand $\overline{B}^A$ is a closed set in A.

It follows by Proposition 1.56 that $\overline{B}^A = F \cap A$, where F is closed in X. Since $B \subseteq \overline{B}^A \subseteq F$ it follows that $\overline{B}^X \subseteq F$, hence $\overline{B}^X \cap A \subseteq F \cap A = \overline{B}^A$.

We conclude that $\overline{B}^X \cap A = \overline{B}^A$.

(b) Since $\mathring{B}^X$ is open in X and contained in $B \subseteq A$, it is an open set in A. Therefore $\mathring{B}^X \subseteq \mathring{B}^A$. □

Remarks 1.58. (1) The inclusion $\mathring{B}^X \subseteq \mathring{B}^A$ *may be strict.*

Consider for instance

$$B = (0,1) \times \{0\} \quad \subset \quad A = \mathbb{R} \times \{0\} \quad \subset \quad X = \mathbb{R}^2.$$

As B contains no open ball in $\mathbb{R}^2$, $\mathring{B}^X$ is empty.

But B is an open set in A, therefore $\mathring{B}^A = B$.

(2) *If A is open then* $\mathring{B}^X = \mathring{B}^A$ for every $B \subseteq A$.

Indeed $\mathring{B}^A$ is open in A, hence $\mathring{B}^A = U \cap A$ for some U open in X.

Since A is also open, it follows that $\mathring{B}^A$ is open in X. It is also contained in B, therefore $\mathring{B}^A \subseteq \mathring{B}^X$.

This and Proposition 1.57, (b), imply equality.

We finish our discussion of subspaces of topological spaces with some easy to see but important remarks.

Remarks 1.59. (1) If $A \subseteq B \subseteq X$ and $(X, \mathcal{T})$ is a topological space *then the topology induced on A by $\mathcal{T}$ coincides with the topology induced on A by $\mathcal{T}_B$.*

(2) If $A \subseteq X$ and (X, d) is a metric space then *the topology on A induced by $\mathcal{T}_{\mathrm{d}}$ coincides with the topology on A induced by the restricted metric d_A.*

This follows immediately from the fact that for any $a \in A$, the ball centred in a and of radius $r > 0$ in (A, d_A) coincides with $B(a,r) \cap A$, where $B(a,r)$ is the ball centred in a and of radius $r > 0$ in (X, d).

(3) If A is open in X then *$\mathcal{T}_A$ is contained in $\mathcal{T}$.*

(4) If A is closed in X then *any set closed in A is closed in X.*

1.8. **Basis for a topology.** As in the case of vector spaces, we can consider a minimal amount of data (a *basis*) via which we may recover the whole space endowed with the considered structure.

This notion is particularly useful in two settings:

- that of topologies induced by metrics;
- that of products of topological spaces (to be studied further on).

Definition 1.60. Given a topological space $(X, \mathcal{T})$, a collection of subsets of X is a *basis* for $\mathcal{T}$ if

(1) $\mathcal{B} \subseteq \mathcal{T}$ (in other words, every set in $\mathcal{B}$ is open), and
(2) every set in $\mathcal{T}$ can be expressed as a union of sets in $\mathcal{B}$.

Example 1.61. In a metric space (X, d) the family of open balls

$$\mathcal{B} = \{B(x,r) : x \in X, \, r > 0\}$$

is a basis for $\mathcal{T}_{\mathrm{d}}$.

In particular in $\mathbb{R}$, the family of open intervals $\mathcal{B} = \{(a,b) \mid a, b \in \mathbb{R}, \, a < b\}$ is a basis for the standard topology.

Moreover, the family of open intervals *with rational endpoints* $\mathcal{B}_{\mathbb{Q}} = \{(a,b) \mid a, b \in \mathbb{Q}, \, a < b\}$ is a basis for the standard topology in $\mathbb{R}$ (Ex. 9, (2), Sheet 1).

Criterion 1.62. *Let $(X, \mathcal{T})$ be a topological space, and $\mathcal{B}$ a basis for $\mathcal{T}$.*

A subset U is open in $(X, \mathcal{T})$ if and only if for every $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Proof. Indeed, if U is open then there exists a collection of subsets $(B_i)_{i \in I}$ in the basis $\mathcal{B}$ such that $U = \bigcup_{i \in I} B_i$. If $x \in U$, then $x \in B_i \subseteq U$, for some $i \in I$.

Conversely assume that for every $x \in U$ there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. Then $U \subseteq \bigcup_{x \in U} B_x \subseteq U$, whence $U = \bigcup_{x \in U} B_x$. So, U is open. $\square$

Proposition 1.63. *Let X, Y be topological spaces, and let $\mathcal{B}$ be a basis for the topology on Y.*

A map $f : X \to Y$ is continuous if and only if for every B in $\mathcal{B}$ its inverse image $f^{-1}(B)$ is open in X.

Proof. The necessary part follows from the fact that $\mathcal{B} \subseteq \mathcal{T}$, i.e. a basis is composed of open sets.

The sufficient part follows from:

- the fact that every open set in Y can be written as $\bigcup_{i \in I} B_i$, with $B_i \in \mathcal{B}$ for every i;
- the fact that $f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i)$.

□

Now we shall see:

- how to recognise when a family of subsets is *a basis of some topology*;
- how to *re-construct the topology* when a basis is given.

Proposition 1.64. *Let X be a set and $\mathcal{B}$ a family of subsets of X such that*

(B1) *X is a union of sets in $\mathcal{B}$;*

(B2) *the intersection $B \cap B'$ of any $B, B' \in \mathcal{B}$ can be expressed as a union of sets in $\mathcal{B}$.*

Then the family $\mathcal{T}_{\mathcal{B}}$ of all unions of sets in $\mathcal{B}$ is a topology for X. Note that $\mathcal{B}$ is a basis for $\mathcal{T}_{\mathcal{B}}$.

Proof. (T1) The set X is contained in $\mathcal{T}_{\mathcal{B}}$ due to (B1).

The empty set $\emptyset$ is contained in $\mathcal{T}_{\mathcal{B}}$ as the union of *no* sets from $\mathcal{B}$.

(T2) Let U, V be two sets in $\mathcal{T}_{\mathcal{B}}$. Then $U = \bigcup_{i \in I} B_i$ and $V = \bigcup_{j \in J} B'_j$, where $B_j, B'_j \in \mathcal{B}$.

The intersection $U \cap V$ is equal to $\bigcup_{i \in I,\, j \in J} B_i \cap B'_j$. According to (B2), $B_i \cap B'_j = \bigcup_{k \in K_{ij}} B''_k$, for some collection $(B''_k)_{k \in K_{ij}}$ in $\mathcal{B}$. It follows that

$$U \cap V = \bigcup_{i \in I,\, j \in J} \bigcup_{k \in K_{ij}} B''_k \,,$$

therefore $U \cap V$ is in $\mathcal{T}_{\mathcal{B}}$.

Property (T3) is immediate from the definition of $\mathcal{T}_{\mathcal{B}}$. □

1.9. **Product of topological spaces.** One of the main ways of *constructing new topological spaces out of given ones* is by taking Cartesian products. This is what we shall discuss now.

Proposition 1.65. *Let $(X,\, \mathcal{T}_X)$, $(Y,\, \mathcal{T}_Y)$ be topological spaces. The family of subsets*

$$\mathcal{B}_{X \times Y} = \{U \times V : U \in \mathcal{T}_X,\, V \in \mathcal{T}_Y\}\,,$$

satisfies the conditions (B1) and (B2) in Proposition 1.64.

Proof. Indeed $X \times Y$ is itself in $\mathcal{B}_{X \times Y}$. This accounts for (B1).

Also, consider $U \times V$ and $U' \times V'$, with $U, U' \in \mathcal{T}_X$ and $V, V' \in \mathcal{T}_Y$. Then

$$(U \times V) \cap (U' \times V') = (U \cap U') \times (V \cap V')$$

is in $\mathcal{B}_{X \times Y}$. This accounts for (B2). □

Definition 1.66. The family of all unions of sets from $\mathcal{B}_{X \times Y}$ is a topology $\mathcal{T}_{X \times Y}$ for $X \times Y$ (according to Proposition 1.64).

We call this topology *the product topology.*

The space $(X \times Y,\, \mathcal{T}_{X \times Y})$ is called *the topological product* of $(X,\, \mathcal{T}_X)$ and $(Y,\, \mathcal{T}_Y)$.

NB One of the commonest errors about products is to assume that **any** open set in the product is a ‘rectangular’ open set such as $U \times V$.

Criterion 1.67. *A subset $W \subseteq X \times Y$ is open in the product topology if and only if for any $(x,\, y) \in W$ there exist $U \in \mathcal{T}_X$, $V \in \mathcal{T}_Y$ such that $(x,\, y) \in U \times V \subseteq W$.*

Proposition 1.68. *Let $(X \times Y, \mathcal{T}_{X\times Y})$ be the topological product of $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$.*

If A is closed in X and B is closed in Y then $A \times B$ is closed in $X \times Y$.

Proof. Since A is closed, its complement $U = X \setminus A$ is open.

Likewise B closed implies $V = Y \setminus B$ open.

The complement $(X \times Y) \setminus (A \times B)$ *is not* $U \times V$, *but*

$$\{(x, y) \in X \times Y \ : \ x \notin A \text{ or } y \notin B\} = (U \times Y) \cup (X \times V)\,.$$

Therefore it is open. □

Remark 1.69. For finitely many topological spaces $X_1, ..., X_n$ one can define a topology on $X_1 \times \cdots \times X_n$ either by considering a basis composed of products of open sets $U_1 \times \cdots \times U_n$, or by induction on n and by identifying $X_1 \times \cdots \times X_n$ with $(X_1 \times \cdots \times X_{n-1}) \times X_n$.

Proposition 1.70. *The product topology on $\mathbb{R}^n$ coincides with the standard topology.*

Proof. We need to show that these two topologies on $\mathbb{R}^n$ have the same open sets.

Recall that the standard topology on $\mathbb{R}^n$ arises from any of the metrics d_1, d_2 and d_∞. These three metrics all give the same topology. In this proof, it is most convenient to use d_∞, which is given by

$$\mathrm{d}_\infty((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max_{1 \leq i \leq n} |x_i - y_i|.$$

Consider a subset W of $\mathbb{R}^n$ that is open in the product topology. Then, for any $(x_1, \ldots, x_n) \in W$, there are open sets $U_1, \ldots, U_n$ in $\mathbb{R}$ such that $(x_1, \ldots, x_n) \in U_1 \times \cdots \times U_n \subseteq W$. Since U_i is open and contains x_i, there is $\epsilon_i > 0$ such that $(x_i - \epsilon_i, x_i + \epsilon_i) \subseteq U_i$. Setting $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$, we deduce that $B_{\mathrm{d}_\infty}((x_1, \ldots, x_n), \epsilon) \subseteq W$. So, W is open with respect to d_∞.

Conversely, suppose that a set W is open with respect to d_∞. Then it is a union of open balls. Each open ball is of the form $(x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon)$. This is a rectangular open set in the product topology. So, W is open in the product topology. □

Proposition 1.71. (1) *If X and Y satisfy the first separation axiom then so does their topological product $X \times Y$.*

(2) *If X and Y are Hausdorff spaces, so is their topological product $X \times Y$.*

Proof. (1) Let $(x, y) \neq (x', y')$. Then either $x \neq x'$ or $y \neq y'$.

Assume that $x \neq x'$. Then there exists U open in X containing x and not x'. It follows that $U \times Y$ contains (x, y) and not (x', y'). The case $y \neq y'$ is similar.

The proof of (2) is done along the same lines, using the fact that

$$U \cap V = \emptyset \ \Rightarrow \ (U \times Y) \cap (V \times Y) = \emptyset\,.$$

□

Proposition 1.72. (1) *When $X \times Y$ is endowed with the product topology $\mathcal{T}_{X\times Y}$, the projection maps $p_X : X \times Y \to X$, $p_X(x, y) = x$, and $p_Y : X \times Y \to Y$, $p_Y(x, y) = y$, are continuous.*

(2) *(*a second way of defining the product topology*) The product topology $\mathcal{T}_{X\times Y}$ is* the coarsest topology making p_X and p_Y continuous*: any topology $\mathcal{T}$ on $X \times Y$ with respect to which p_X and p_Y are continuous contains $\mathcal{T}_{X\times Y}$.*

Proof. (1) Indeed for any open set U in X, $p_X^{-1}(U) = U \times Y$; likewise for any open set V in Y, $p_Y^{-1}(V) = X \times V$.

(2) According to the above any topology $\mathcal{T}$ with respect to which p_X and p_Y are continuous must contain all sets $U \times Y$ with $U \in \mathcal{T}_X$, and all sets $X \times V$ with $V \in \mathcal{T}_Y$.

Therefore, by property (T2), $\mathcal{T}$ must contain all sets $U \times V = (U \times Y) \cap (X \times V)$. Thus $\mathcal{B}_{X\times Y} \subseteq \mathcal{T}$ whence, by property (T3) of $\mathcal{T}$, $\mathcal{T}_{X\times Y} \subseteq \mathcal{T}$. □

Proposition 1.73. *Let X, Y be two topological spaces.*

(1) *Let Z be a topological space, and let $f : Z \to X$ and $g : Z \to Y$ be two maps.*

The map $(f, g) : Z \to X \times Y$ is continuous, with $X \times Y$ endowed with the product topology, if and only if both f and g are continuous.

(2) *(*a third way of defining the product topology*) The product topology is the only topology for which (1) holds for all possible Z and f, g.*

Remark 1.74. Proposition 1.73, (1) can be reformulated as follows:

A map $F : Z \to X \times Y$ from a topological space Z into the topological product $X \times Y$ is continuous if and only if both $p_X \circ F : Z \to X$ and $p_Y \circ F : Z \to Y$ are continuous.

Proof of Proposition 1.73. (1) Assume that the map (f, g) is continuous. Then $f = p_X \circ (f, g)$ and $g = p_Y \circ (f, g)$ are also continuous, by Proposition 1.22.

Conversely, assume that both f and g are continuous.

For every U open in X and V open in Y,

$$(f, g)^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V)$$

is open.

This and Proposition 1.63 applied to the map (f, g) and the basis $B_{X\times Y}$ allow us to conclude that (f, g) is continuous.

(2) Let $\mathcal{T}$ be another topology on $X \times Y$ for which (1) holds for all possible Z and f, g.

We take $Z = X \times Y$ with the product topology $\mathcal{T}_{X\times Y}$ and the maps $f = p_X$ and $g = p_Y$ continuous. Then $(p_X, p_Y) = \mathrm{id}_{X\times Y}$ is a continuous map between $(X \times Y, \mathcal{T}_{X\times Y})$ and $(X \times Y, \mathcal{T})$. It follows that for every open set $U \in \mathcal{T}$, its inverse image by $\mathrm{id}_{X\times Y}$, which is U, is contained in $\mathcal{T}_{X\times Y}$.

We have thus proved that $\mathcal{T} \subseteq \mathcal{T}_{X\times Y}$.

Clearly $\mathrm{id}_{X\times Y} = (p_X, p_Y)$ is a continuous map between $(X \times Y, \mathcal{T})$ and $(X \times Y, \mathcal{T})$. This and the property (1) of $\mathcal{T}$ implies that p_X and p_Y are continuous also when $X \times Y$ is endowed with the topology $\mathcal{T}$.

Proposition 1.72, (2), implies that $\mathcal{T}_{X\times Y} \subseteq \mathcal{T}$, whence equality. □

1.10. **Disjoint unions.**

Definition 1.75. Let X and Y be sets.

Their *disjoint union* is
$$X \sqcup Y = (X \times \{0\}) \cup (Y \times \{1\}).$$
Thus, it consists of consists of two types of ordered pairs:
$(x, 0)$, where $x \in X$, and $(y, 1)$, where $y \in Y$.

The point here is that X and Y may not be disjoint. They may even be equal. But, in this construction, we create a copy of X and a copy of Y, which are now disjoint.

When X and Y are topological spaces, their disjoint union has a natural topology:

A set U is open in $X \sqcup Y$ if and only if its intersection with $X \times \{0\}$ is open and its intersection with $Y \times \{1\}$ is open in Y.

1.11. **Connected spaces.** Connected metric spaces were defined in the Metric Spaces course. The definition naturally generalised to topological spaces.

Definition 1.76. A topological space X is *disconnected* if there are disjoint open non-empty subsets U and V such that $U \cup V = X$.

If X is not disconnected, it is called *connected*.

The theorems that were proved about connectedness and path-connectedness in the Metric Spaces course all generalise to topological spaces. We record the results and proofs here, but most will be omitted from the lectures.

Proposition 1.77. *Let X be a topological space. The following properties are equivalent:*

(1) *the only subsets of X which are both open and closed are X and $\emptyset$;*
(2) *X is connected;*
(3) *any continuous map from X to $\{0, 1\}$ (with the discrete topology) is constant.*

Proof. (1)$\Rightarrow$ (2) If U, V are open sets in X such that $U \cap V = \emptyset$ and $U \cup V = X$, then U is both open and closed (as $X \setminus U = V$). It follows by (1) that either $U = \emptyset$ or $U = X$, hence $V = \emptyset$.

(2)$\Rightarrow$ (3) Let $f : X \to \{0, 1\}$ be a continuous map. Then $U = f^{-1}(\{0\})$ and $V = f^{-1}(\{1\})$ are open, disjoint subsets of X, and $U \cup V = X$. So, by definition of connectedness, one of U and V is empty, and therefore f is not surjective.

(3)$\Rightarrow$ (1) Let A be a subset of X both open and closed. Then the map $f : X \to \{0, 1\}$ such that $f(x) = 1$ if $x \in A$ and $f(x) = 0$ if $x \notin A$ is continuous: $f^{-1}(\{1\}) = A$ which is open, $f^{-1}(\{0\}) = X \setminus A$, which is open.

According to (3) either f is constant 1, which means that $A = X$, or f is constant 0, which means that $A = \emptyset$. □

Definition 1.78. A (non-empty) subset A of a topological space X is *connected* if A with the subspace topology is connected.

Conventionally an empty set is assumed to be connected.

Remark 1.79. The connectedness of a subset A can be formulated in terms of the topology on X (rather than the subspace topology) as follows:

A is connected if and only if, whenever U and V are open subsets of X such that $A \subseteq (U \cup V)$ and $U \cap V \cap A = \emptyset$, then either $U \cap A = \emptyset$ or $V \cap A = \emptyset$.

Proposition 1.80. *If $f : X \to Y$ is a continuous map of topological spaces X, Y and $A \subset X$ is connected then so is $f(A)$. ("**The continuous image of a connected set is connected.**")*

Proof. Let U and V be open sets in Y such that $f(A) \subseteq U \cup V$ and $f(A) \cap U \cap V = \emptyset$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are open in X. So, $f^{-1}(U) \cap A$ and $f^{-1}(V) \cap A$ are disjoint and open in A. Since A is connected, one of $f^{-1}(U) \cap A$ and $f^{-1}(V) \cap A$ is empty. Hence, one of $U \cap f(A)$ and $V \cap f(A)$ is empty. So, $f(A)$ is connected. $\square$

Proposition 1.81. *Suppose that $\{A_i : i \in I\}$ is a family of connected subsets of a topological space X with $A_i \cap A_j \neq \emptyset$ for each pair $i, j \in I$.*

Then $\bigcup_{i \in I} A_i$ is connected.

Proof. Let $f : \bigcup_{i \in I} A_i \to \{0, 1\}$ be a continuous map. Its restriction to A_i, $f|_{A_i}$, is also continuous. Since A_i is connected it follows that $f|_{A_i}$ is constant equal to $c_i \in \{0, 1\}$.

For each pair $i, j \in I$ there exists a point x in $A_i \cap A_j$, and $f(x) = c_i = c_j$. Therefore all the constants c_i are equal, and f is constant. $\square$

Corollary 1.82. *If $\{C_i : i \in I\}$ and B are connected subsets of a topological space X such that $C_i \cap B \neq \emptyset$ for each $i \in I$, then $B \cup \bigcup_{i \in I} C_i$ is connected.*

Proof. Proposition 1.81 implies that $A_i = C_i \cup B$ is connected for every $i \in I$.

Since $A_i \cap A_j$ contains B for every i, j, again by Proposition 1.81 it follows that $\bigcup_{i \in I} A_i = B \cup \bigcup_{i \in I} C_i$ is connected. $\square$

Theorem 1.83. *The topological product $X \times Y$ is connected if and only if both X and Y are connected.*

Remark 1.84. An inductive argument allows one to extend Theorem 1.83 to any finite product of topological spaces: a topological product $X_1 \times \cdots \times X_n$ is connected if and only if $X_1, ..., X_n$ are connected.

Proof of Theorem 1.83. If $X \times Y$ is connected then $X = p_X(X \times Y)$ and $Y = p_Y(X \times Y)$ are connected by Proposition 1.80.

Assume that X and Y are connected. Then $X \times \{y\}$ is connected for every $y \in Y$, and $\{x\} \times Y$ is connected for every $x \in X$.

Fix a point $y_0 \in Y$. Corollary 1.82 applied to $B = X \times \{y_0\}$ and to $C_x = \{x\} \times Y$, connected and such that $B \cap C_x = \{(x, y_0)\}$ implies that $B \cup \bigcup_{x \in X} C_x = X \times Y$ is connected. $\square$

Theorem 1.85. *Suppose that A is a connected subset of a topological space X and $A \subseteq B \subseteq \overline{A}$. Then B is connected.*

Proof. Let U and V be open subsets of X such that $B \subseteq U \cup V$ and $B \cap U \cap V = \emptyset$. We will show that one of $U \cap B$ and $V \cap B$ is empty.

Now, $A \subseteq U \cup V$ and $A \cap U \cap V \subseteq B \cap U \cap V = \emptyset$. So, by the connectedness of A, one of $U \cap A$ and $V \cap A$ is empty. Say that $U \cap A$ is empty. Then, A lies in the closed set $X \backslash U$. So, by Proposition 1.28, $\overline{A} \subseteq X \backslash U$. So, $U \cap \overline{A} = \emptyset$. Since $B \subseteq \overline{A}$, we deduce that $U \cap B$ is empty, as required. $\square$

Definition 1.86. A *path connecting two points* x, y in a topological space X is a continuous map $\mathfrak{p} : [0, 1] \to X$ with $\mathfrak{p}(0) = x$, $\mathfrak{p}(1) = y$.

Remark 1.87. It was shown in the Metric Spaces course that any interval in $\mathbb{R}$ is connected. Hence, by Proposition 1.80, if $\mathfrak{p} : [0,\,1] \to X$ is a path then $\mathfrak{p}\,([0,\,1])$ is a connected set.

Definition 1.88. A topological space X is *path-connected* if any two points in X are connected by a path in X.

A subset $A \subseteq X$ is *path-connected* if with the subspace topology it satisfies the previous condition. Equivalently, A is path-connected if any two points in A can be joined by a path $\mathfrak{p}\colon [0,1] \to X$ with image in A. Conventionally the empty set is path-connected.

Proposition 1.89. *Any path-connected space is connected.*

Proof. Let X be a path-connected space and let a be a fixed point in X. For every $x \in X$ there exists a path $\mathfrak{p}_x : [0,1] \to X$ such that $\mathfrak{p}_x(0) = a$ and $\mathfrak{p}_x(1) = x$. By Remark 1.87, $\mathcal{P}_x = \mathfrak{p}_x[0,1]$ is a connected set.

We apply Corollary 1.82 to the collection of connected sets $\{\mathcal{P}_x : x \in X\}$ and to the connected set $\{a\}$, and conclude that $\bigcup_{x\in X} \mathcal{P}_x \cup \{a\} = X$ is connected. □

Remark 1.90. **There exist connected spaces that are not path-connected.** An example was given in the Metric Spaces course.

Proposition 1.91. *If $f : X \to Y$ is a continuous map of topological spaces X, Y and $A \subset X$ is path-connected then so is $f(A)$. ("**The continuous image of a path-connected set is path-connected.**")*

Proof. For every two elements $x, y \in f(A)$ let a, b be two elements in A such that $f(a) = x$ and $f(b) = y$.

The set A is path-connected, therefore there exists a path $\mathfrak{p} : [0,1] \to A$ connecting a and b. Then $f \circ \mathfrak{p}$ is a path in $f(A)$ connecting x and y. □

2. Compact spaces

Compact sets are a particular type of bounded closed set, very much used in all branches of mathematics. Here are two reasons why this notion is interesting:

- in Analysis it is important to know whether there exists a certain object minimizing a given functional or not (for instance in calculus of variations problems);
- from the topological point of view, as it will become clear from the theory, compact sets are 'the next best thing, after singletons and finite sets'.

2.1. Definition and properties of compact spaces.

Definition 2.1. A family $\{U_i : i \in I\}$ of subsets of a space X is called a *cover* if $X = \bigcup\limits_{i\in I} U_i$.

If each U_i is open in X then $\mathcal{U}$ is called *an open cover* for X.

Definition 2.2. A *subcover* of a cover $\{U_i : i \in I\}$ for a space X is a subfamily $\{U_j : j \in J\}$ for some subset $J \subseteq I$ such that $\{U_j : j \in J\}$ is still a cover for X.

We call it a *finite (or countable) subcover* if J is finite (or countable).

Definition 2.3. A topological space X is *compact* if any open cover of X has a finite subcover.

Proposition 2.4. *Let X be a topological space. The following are equivalent:*

(1) *X is compact;*

(2) *if $\{V_i : i \in I\}$ is an indexed family of closed subsets of X such that $\bigcap_{j\in J} V_j \neq \emptyset$ for any finite subset $J \subseteq I$ then $\bigcap_{i\in I} V_i \neq \emptyset$.*

Proof. (1) $\Rightarrow$ (2) Assume that $\{V_i : i \in I\}$ is a family of closed subsets of X such that $\bigcap_{j\in J} V_j \neq \emptyset$ for any finite subset $J \subseteq I$, while $\bigcap_{i\in I} V_i = \emptyset$. Then $\bigcup_{i\in I}(X \setminus V_i) = X$.

According to (1) there exists J finite subset in I such that $\bigcup_{j\in J}(X \setminus V_j) = X$. This is equivalent to $\bigcap_{j\in J} V_j = \emptyset$, contradicting the hypothesis.

(2) $\Rightarrow$ (1) Let $\{U_i : i \in I\}$ be an open cover of X. Then $X = \bigcup_{i\in I} U_i$, whence $\bigcap_{i\in I}(X \setminus U_i) = \emptyset$.

According to property (2) applied to the family of closed sets $\{X \setminus U_i \ : \ i \in I\}$ there exists some finite subset J of I such that $\bigcap_{j\in J}(X \setminus U_j) = \emptyset$. Equivalently $\bigcup_{j\in J} U_j = X$. □

Definition 2.5. A subset A of a topological space X is *compact* if it is compact when endowed with the subspace topology.

Remark 2.6. In view of the way in which the subspace topology is defined, a subset A of a topological space X is compact if and only if for every family $\{U_i : i \in I\}$ of *open sets in* X such that $A \subseteq \bigcup_{i\in I} U_i$ there exists a finite subset $J \subseteq I$ such that $A \subseteq \bigcup_{j\in J} U_j$.

This can be rephrased in terms of open covers for A, as follows.

Definition 2.7. A family $\{U_i : i \in I\}$ of subsets of a space X is called a *cover for* A if $A \subseteq \bigcup_{i\in I} U_i$.

We also sometimes say that the sets U_i, $i \in I$, *cover the set* A.

Thus, a subset A of X is compact if and only if every family of open sets in X which form a cover for A have a finite subcover.

Examples 2.8. (1) *Any finite space X is compact.*

(2) *If $(x_n)_{n\in\mathbb{N}}$ is a sequence converging to x in a topological space then the set*

$$L = \{x_n \mid n \in \mathbb{N}\} \cup \{x\}$$

is compact.

Indeed consider $\{U_i : i \in I\}$, an open cover of L. There exists $k \in I$ such that $x \in U_k$.

The sequence (x_n) converges to x therefore there exists N such that $x_n \in U_k$ for every $n \geq N$. For every $n \in \{1, ..., N\}$, x_n is contained in some U_{i_n}, $i_n \in I$. Therefore L is contained in $U_{i_1} \cup \cdots \cup U_{i_N} \cup U_k$.

(3) *Any (non-empty) set with the cofinite topology is compact.* Likewise for the indiscrete topology. Thus *any set X can be endowed with a topology $\mathcal{T}$ such that $(X, \mathcal{T})$ is compact.*

The statement for the indiscrete topology follows from the fact that there are only two open sets in that topology.

The statement for the co-finite topology is easier to check with Proposition 2.4, (2). Indeed consider a family $\{F_i : i \in I\}$ of closed sets in X such that any intersection of finitely many of those sets is non-empty.

Pick $k \in I$ and consider $F_k = \{x_1, ..., x_m\}$. Assume that for every x_j there exists F_{i_j} not containing it. Then $F_k \cap F_{i_1} \cap \cdots \cap F_{i_m}$ is empty. This contradicts the hypothesis.

Therefore some x_j is contained in all F_i with $i \in I$. It follows that $\bigcap_{i \in I} F_i$ contains x_j.

The following was proved in the Metric Spaces course.

Proposition 2.9. *(the Heine-Borel Theorem) Any interval $[a, b]$ in $\mathbb{R}$ is compact.*

Remark 2.10. An open interval (a, b), where $a < b$, is not compact in $\mathbb{R}$ (with the Euclidean topology). Nor are the intervals $[a, \infty)$ or $(-\infty, b]$.

Indeed $(a, b) = \bigcup_{n \in \mathbb{N}} \left(a + \frac{1}{n}\, ,\, b - \frac{1}{n}\right)$. Any finite subfamily $\left(a + \frac{1}{n_1}\, ,\, b - \frac{1}{n_1}\right)$,......, $\left(a + \frac{1}{n_k}\, ,\, b - \frac{1}{n_k}\right)$ (for which we may assume that $n_1 < ... < n_k$, otherwise we change the order) covers only $\left(a + \frac{1}{n_k}\, ,\, b - \frac{1}{n_k}\right)$.

A similar argument can be made for $[a, \infty) = \bigcup_{n \in \mathbb{N}} [a, n)$ and for $(-\infty, b] = \bigcup_{n \in \mathbb{N}} (-n, b]$.

Proposition 2.11. *Any closed subset A of a compact space X is compact.*

Proof. Indeed let $\{V_i : i \in I\}$ be an indexed family of closed subsets of A such that $\bigcap_{j \in J} V_j \neq \emptyset$ for any finite subset $J \subseteq I$.

As A is closed in X, V_i are also closed in X, which is compact. Therefore $\bigcap_{i \in I} V_i \neq \emptyset$. □

Remark 2.12. *The converse of Proposition 2.11 may not be true*: a singleton in X is always compact but may not be closed.

As the example suggest, **a separation axiom must be added** in order to have the converse of Proposition 2.11.

Proposition 2.13. *Let X be a Hausdorff space.*

If K is a compact subset of X and $x \in X \setminus K$ then there exists U, V disjoint open sets such that $K \subseteq U$ and $x \in V$.

Proof. For every $y \in K$, according to the Hausdorff property, there exists a pair of disjoint open sets U_y and V_y, $y \in U_y$, $x \in V_y$.

The collection $\{U_y : y \in K\}$ is an open cover for K. Therefore there exist $y_1, ..., y_m$ such that $K \subseteq U_{y_1} \cup \cdots \cup U_{y_m}$. Let $U = U_{y_1} \cup \cdots \cup U_{y_m}$.

The set $V = V_{y_1} \cap \cdots \cap V_{y_m}$ is open, it contains x, and $V \cap U = V \cap \bigcup_{j=1}^{m} U_{y_j} = \emptyset$. □

Corollary 2.14. *Any compact subset K of a Hausdorff space X is closed in X.*

Proof. According to Proposition 2.13, for every $x \in X \setminus K$, there exists V open such that $x \in V \subseteq X \setminus K$. This implies that $X \setminus K$ is open, by Lemma 1.26. □

Proposition 2.15. *Let X be a topological space.*

(1) *If $K_1, ..., K_n$ are compact subsets in X then $K_1 \cup \cdots \cup K_n$ is compact in X.*
(2) *If $\{K_i : i \in I\}$ is a non-empty family of compact subsets and X **is Hausdorff** then $\bigcap_{i\in I} K_i$ is compact.*

Proof. (1) Let $\{U_i : i \in I\}$ be an open cover of $K_1 \cup \cdots \cup K_n$. In particular for every $t \in \{1, ..., n\}$, $K_t \subseteq \bigcup_{i\in I} U_i$. As K_t is compact, there exists a finite subset J_t of I such that $K_t \subseteq \bigcup_{j\in J_t} U_j$.

It follows that $K_1 \cup \cdots \cup K_n \subseteq \bigcup_{t=1}^{n} \bigcup_{j\in J_t} U_j$.

(2) Since X is Hausdorff, every K_i is a closed set according to Corollary 2.14. Therefore $L = \bigcap_{i\in I} K_i$ is also closed. It is also contained in some (each) compact space K_i, hence by Proposition 2.11, L is compact. □

Remark 2.16. (1) Property (1) is no longer true for infinite unions.

Example: $\bigcup_{n\in\mathbb{N}} [0, n] = [0, \infty)$.

(2) Property (2) is no longer true if X is not Hausdorff (see Ex. 4, (2), Sheet 3).

Proposition 2.17. *A compact subset A of a metric space X is bounded.*

Proof. Let x be a point in X. We have that $A \subseteq X = \bigcup_{n\in\mathbb{N}} B(x, n)$.

It follows that there exist $n_1 < ... < n_k$ such that $A \subseteq B(x, n_1) \cup \cdots \cup B(x, n_k) = B(x, n_k)$. □

Corollary 2.18. *A compact subset of a metric space Y is bounded and closed in Y.*

2.2. Compact spaces and continuous maps.

Proposition 2.19. *If $f : X \to Y$ is a continuous map of topological spaces and $A \subset X$ is compact then so is $f(A)$. (‘**The continuous image of a compact set is compact.**’)*

Proof. Let $\{U_i : i \in I\}$ be an open cover of $f(A)$.

Then $\{f^{-1}(U_i) : i \in I\}$ is an open cover of A.

The compactness of A implies that there exists a finite subcover $\{f^{-1}(U_j) : j \in J\}$, $J \subseteq I$.

It follows that $\{U_j : j \in J\}$ is a finite subcover for $f(A)$. □

Corollary 2.20. *Suppose that X is a compact space, Y is a metric space and $f : X \to Y$ is continuous.*

Then $f(X)$ is bounded and closed in Y. (When $f(X)$ is bounded we say ‘f is bounded’.)

Corollary 2.21 (the extreme value theorem)**.** *If $f : X \to \mathbb{R}$ is a continuous real-valued function on a compact space X then f is bounded and attains its bounds, i.e. there exist $m, M \in X$ such that $f(m) \le f(x) \le f(M)$ for every $x \in X$.*

Proof. Since X is compact, $f(X)$ is compact, in particular it is bounded and closed.

Let a, A be the highest lower bound, respectively the least upper bound of $f(X)$.

Since $f(X)$ is closed in $\mathbb{R}$, it follows that both a and A are in $f(X)$. □

Remark 2.22. Another consequence of Proposition 2.19 is the following: if $A \subset X \subset Y$, where Y is a topological space and X is endowed with the subspace topology, then A compact in $X \Rightarrow A$ compact in Y. This follows by using Proposition 2.19 for the inclusion map $i : X \to Y$.

The same implication *does not hold* when 'compact' is replaced by 'closed': $(0,1]$ is closed in $(0,2)$ but not in $\mathbb{R}$.

The converse implication holds both for 'closed' and for 'compact', i.e. A compact (closed) in $Y \Rightarrow A$ compact (closed) in X.

Theorem 2.23. *The product $X \times Y$ is compact if and only if both X and Y are compact.*

Proof. If the product $X \times Y$ is compact then $X = p_X(X \times Y)$ and $Y = p_Y(X \times Y)$ are compact, due to Proposition 2.19.

Conversely, assume that X and Y are compact.

Let $\{W_i : i \in I\}$ be an open cover of $X \times Y$.

Since the rectangular open sets $U \times V$ compose a basis for the product topology, we may write

$$X \times Y = \bigcup_{i \in I} W_i = \bigcup_{j \in J} U_j \times V_j \,,$$

where each $U_j \times V_j$ is contained in some W_i.

Therefore it is enough if we prove that the open cover $\{U_j \times V_j : j \in J\}$ has a finite subcover.

For every $y \in Y$, the compact set $X \times \{y\}$ is covered by $\{U_j \times V_j : j \in J\}$. Therefore there exists a finite subset $F_y \subseteq J$ such that

$$X \times \{y\} \subseteq \bigcup_{j \in F_y} U_j \times V_j \,.$$

The set $V_y = \bigcap_{j \in F_y} V_j$ is an open set containing y. The family $\{V_y : y \in Y\}$ is an open cover of Y, which is compact.

It follows that there exist $y_1, ..., y_m$ such that $Y = V_{y_1} \cup \cdots \cup V_{y_m}$.

We state that $X \times Y = \bigcup_{k=1}^m \bigcup_{j \in F_{y_k}} U_j \times V_j$. Indeed consider an arbitrary element (x,y) in the product. Then there exists $k \in \{1, ..., m\}$ such that $y \in V_{y_k}$. In particular $y \in V_j$ for all $j \in F_{y_k}$. On the other hand $X \times \{y_k\} \subseteq \bigcup_{j \in F_{y_k}} U_j \times V_j$, whence there exists $j_0 \in F_{y_k}$ such that $x \in U_{j_0}$. It follows that $(x,y) \in U_{j_0} \times V_{j_0}$. □

Theorem 2.24. ***(general Heine-Borel)*** *Any closed bounded subset of $(\mathbb{R}^n, \| \ \|_i)$, where $i \in \{1, 2, \infty\}$, is compact.*

Proof. Let A be a closed bounded subset of $\mathbb{R}^n$. Since A is bounded, there exists $\overline{x} = (x_1, ..., x_n)$ and $r > 0$ such that $A \subseteq B_{\mathrm{d}_\infty}(\overline{x}, r)$.

The ball $B_{\mathrm{d}_\infty}(\overline{x}, r)$ in $(\mathbb{R}^n, \| \ \|_\infty)$ can be rewritten as

$$B_{\mathrm{d}_\infty}(\overline{x}, r) = (x_1 - r, x_1 + r) \times \cdots \times (x_n - r, x_n + r) \subseteq [x_1 - r, x_1 + r] \times \cdots \times [x_n - r, x_n + r] \,.$$

Each interval $[x_i - r, x_i + r]$ is compact by Proposition 2.9, and their product is compact by Theorem 2.23.

It follows that A is a closed subset contained in a compact space, therefore it is compact by Proposition 2.11. $\square$

N.B. Theorem 2.24 fails in general when $(\mathbb{R}^n, \| \|_i)$ is replaced by an arbitrary metric space (X, d):

- an easy example of this is $(0, 1]$ - it's bounded, and closed in $(0, 2)$, but it's not compact;
- another such example, *in which the ambient space is a normed vector space, moreover complete,* will be given in Exercise 2, Sheet 3.

The inverse of a continuous bijective map is not necessarily continuous. Still, the statement is true under some extra conditions.

Theorem 2.25. *Suppose that X is a compact space, that Y is a Hausdorff space and that $f : X \to Y$ is a continuous bijection. Then f is a homeomorphism (i.e. f^{-1} is continuous too).*

Proof. Let V be a closed subset of X. Then V is compact, by Proposition 2.11. It follows that $f(V)$ is compact, by Proposition 2.19. Since Y is Hausdorff, this implies that $f(V)$ is closed, according to Corollary 2.14. Hence, f^{-1} is continuous. $\square$

Corollary 2.26. *Suppose that X is a compact space, that Y is a Hausdorff space and that $f : X \to Y$ is injective and continuous.*

Then f determines a homeomorphism of X onto $f(X)$.

Remark 2.27. This generalises the Prelims result: if $f : [0, 1] \to \mathbb{R}$ is continuous and monotonic then its inverse function $f^{-1} : f([0, 1]) \to [0, 1]$ is continuous.

2.3. **Sequential compactness.** We already saw that **in a metric space** *many topological properties can be characterised in terms of sequences.*

In what follows we show that one of these properties is compactness.

Definition 2.28. A topological space X is *sequentially compact* if every sequence in X has a subsequence converging to a point in X.

A (non-empty) subset A of a topological space X is *sequentially compact* if, with the subspace topology, A is sequentially compact. (Conventionally, the empty set is sequentially compact.)

Theorem 2.29. (1) *(Bolzano-Weierstrass theorem) In a compact* ***topological*** *space X every infinite subset has accumulation points.*

(2) *Every compact* ***metric*** *space is sequentially compact.*

Proof. (1) We prove that if, for a subset A of X, the set of accumulation points A' is empty then A is finite.

Recall that the closure $\overline{A}$ is equal to $A \cup A'$. Therefore if A' is empty then $\overline{A} = A = A \setminus A'$.

The set $A \setminus A'$ is *the set of isolated points of A, hence all points of A are isolated:* for every $a \in A$ there exists U_a open set such that $U_a \cap A = \{a\}$.

We have that $\overline{A} = A \subseteq \bigcup_{a \in A} U_a$. The set $\overline{A}$ is a closed set in a compact topological space, therefore it is compact by Proposition 2.11. Hence, there exist $a_1, ..., a_n$ in A such that $\overline{A} = A \subseteq \bigcup_{i=1}^n U_{a_i}$.

We may then write $A = \bigcup_{i=1}^{n} (U_{a_i} \cap A) = \bigcup_{i=1}^{n} \{a_i\} = \{a_1, ..., a_n\}$.

(2) Consider (x_n) a sequence in a compact metric space X. Let $A = \{x_n : n \in \mathbb{N}\}$.

Assume that A is finite, $A = \{a_1, ..., a_k\}$. For every a_i we consider the set $\mathbb{N}_i = \{n \in \mathbb{N} : x_n = a_i\}$. Since $\mathbb{N} = \mathbb{N}_1 \sqcup \mathbb{N}_2 \sqcup \cdots \sqcup \mathbb{N}_k$, at least one of the sets $\mathbb{N}_i$ is infinite. It follows that (x_n) has a subsequence which is constant, therefore convergent.

Assume that A is infinite. Then according to (1), $A' \neq \emptyset$. Let a be a point in A'. For every open set U containing a, $(U \setminus \{a\}) \cap A \neq \emptyset$.

In particular there exists

- a term x_{n_1} of the sequence contained in $[B(a, 1) \setminus \{a\}] \cap A$,
- a term x_{n_2} in $\left[B\left(a, \frac{1}{2}\right) \setminus \{a, x_1, ..., x_{n_1}\}\right] \cap A$, etc.
- a term x_{n_k} in $\left[B\left(a, \frac{1}{k}\right) \setminus \{a, x_1, ..., x_{n_{k-1}}\}\right] \cap A$.

In other words, we construct inductively a subsequence of (x_n) converging to a. □

Theorem 2.30. *Any compact metric space is complete.*

Proof. Let (x_n) be a Cauchy sequence in a compact metric space (X, d).

According to Theorem 2.29, (2), (x_n) has a subsequence $(x_{\varphi(n)})$ convergent to a point a, where $\varphi : \mathbb{N} \to \mathbb{N}$ is a one-to-one increasing map. We prove that (x_n) converges to a.

Consider an arbitrary $\varepsilon > 0$. The sequence (x_n) is Cauchy, therefore there exists N such that for every $n, m \geq N$, $\mathrm{d}(x_n, x_m) < \frac{\varepsilon}{2}$.

The subsequence $(x_{\varphi(n)})$ converges to a, therefore there exists M such that for every $n \geq M$, $\mathrm{d}\left(x_{\varphi(n)}\,,\, a\right) < \frac{\varepsilon}{2}$.

Let $k \geq M$ large enough so that $\varphi(k) \geq N$. Then for every $n \geq N$,

$$\mathrm{d}(x_n, a) \leq \mathrm{d}\left(x_n\,,\, x_{\varphi(k)}\right) + \mathrm{d}\left(x_{\varphi(k)}\,,\, a\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

□

Theorem 2.31. *Any sequentially compact metric space is compact.*

N.B. This and Theorem 2.29, (2), imply that **a metric space is compact if and only if it is sequentially compact**.

The proof is done in two steps, which we formulate as two separate Propositions.

Proposition 2.32. *Let X be a sequentially compact metric space. For any open cover $\mathcal{U}$ of X there exists $\varepsilon > 0$ such that for every $x \in X$, $B(x, \varepsilon)$ is entirely contained in some single set from $\mathcal{U}$.*

Definition 2.33. The number ε in Proposition 2.32 is called a *Lebesgue number for the cover $\mathcal{U}$*.

Proof. Assume that for every $n \in \mathbb{N}$ there exists a point $x_n \in X$ such that the ball $B\left(x_n\,,\,\frac{1}{n}\right)$ is contained in no subset $U \in \mathcal{U}$.

The sequence (x_n) has a subsequence $(x_{\varphi(n)})$ converging to some point a. There exists a subset U in the cover $\mathcal{U}$ containing a. As U is open, it also contains a ball $B(a, 2\varepsilon)$, for $\varepsilon > 0$.

The subsequence $(x_{\varphi(n)})$ converges to a, hence there exists N such that if $n \geq N$ then $x_{\varphi(n)} \in B(a, \varepsilon)$. This implies that $B\left(x_{\varphi(n)}, \varepsilon\right) \subseteq B(a, 2\varepsilon) \subseteq U$.

For n large enough $\frac{1}{\varphi(n)} < \varepsilon$, whence $B\left(x_{\varphi(n)}, \frac{1}{\varphi(n)}\right) \subseteq B\left(x_{\varphi(n)}, \varepsilon\right) \subseteq U$.

The last inclusion contradicts the choice of the sequence (x_n). □

Definition 2.34. Given a real number $\varepsilon > 0$, an *ε-net* for a metric space X is a subset $N \subseteq X$ such that $\{B(x, \varepsilon) : x \in N\}$ is a cover of X.

Example 2.35. For every $\varepsilon > \frac{\sqrt{n}}{2}$, $\mathbb{Z}^n$ is an ε-net for $\mathbb{R}^n$ with the Euclidean norm $\|\ \|_2$.

Proposition 2.36. *Let X be a sequentially compact metric space and let ε be an arbitrary positive number.*

Then there exists a finite ε-net for X.

Remark 2.37. Thus *compact metric spaces can be approximated by finite sets.*

This illustrates once more our remark that from the topological point of view compact sets are 'the next best thing, after singletons and finite sets'.

Proof. Assume that there exists no finite ε-net for X.

Let x_1 be a point in X. According to the assumption there exists $x_2 \in X \setminus B(x_1, \varepsilon)$.

We argue by induction and assume that there exist $x_1, ..., x_n$ points in X such that $\mathrm{d}(x_i, x_j) \geq \varepsilon$ for every $i \neq j$ in $\{1, 2, ..., n\}$.

Since $\{x_1, ..., x_n\}$ is finite, it cannot be a ε-net. Therefore there exists a point x_{n+1} in $X \setminus \bigcup_{i=1}^n B(x_i, \varepsilon)$.

In this manner we construct a sequence (x_n) such that

$$\mathrm{d}(x_i, x_j) \geq \varepsilon \ \ \forall i \neq j,\ i, j \in \mathbb{N}. \tag{1}$$

The space X is sequentially compact, therefore (x_n) has a convergent subsequence $(x_{\varphi(n)})$, where $\varphi : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function.

In particular $(x_{\varphi(n)})$ is a Cauchy sequence. It follows that there exists N such that if $m \geq n \geq N$, $\mathrm{d}(x_{\varphi(n)}, x_{\varphi(m)}) < \varepsilon$. This contradicts (1). □

We now finish the proof of the fact that a sequentially compact metric space is compact.

Indeed, let $\mathcal{U}$ be an open cover of X and let ε be the Lebesgue number for the cover $\mathcal{U}$ provided by Proposition 2.32.

Proposition 2.36 implies that there exists a finite ε-net, $\{x_1, ..., x_n\}$.

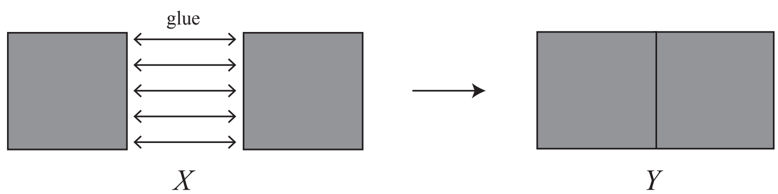
By the definition of the Lebesgue number, each ball $B(x_i, \varepsilon)$ is contained in some set $U_i \in \mathcal{U}$.

Thus $X = \bigcup_{i=1}^n B(x_i, \varepsilon) \subseteq \bigcup_{i=1}^n U_i$. □

3. Quotient spaces

In topology, one often wants to build spaces by starting from simple pieces and 'gluing' them together. The precise construction uses 'quotient spaces'.

Example 3.1. Let $X = ([0,1] \times [0,1]) \cup ([2,3] \times [0,1])$. In other words, X is two disjoint solid squares in the plane. Suppose I want to glue these squares together to form a rectangle Y, as shown below.



We say that certain points of X are 'related' via a relation $\mathcal{R}$. For example, the points $(1,0)$ and $(2,0)$ are related in the above example.

Note that if a, b and c are points in X, and a is glued to b, and b is glued to c, then a must be glued to c. In other words,

$$a\mathcal{R}b \text{ and } b\mathcal{R}c \Rightarrow a\mathcal{R}c.$$

So, $\mathcal{R}$ should be a transitive relation. It should also be reflexive and symmetric. So, $\mathcal{R}$ should be an equivalence relation.

Each equivalence class in X becomes a single point in Y.

So, the key ingredients of the above construction are:

- A topological space X.
- An equivalence relation $\mathcal{R}$ on X.
- A new set Y where there is a single point of Y for each equivalence class of X.

3.1. **Definitions.**

The above discussion used some terminology that was introduced in Prelims. We now recall the various definitions of these terms.

Let X be a set. Recall that a *relation* on X is a subset $\mathcal{R}$ of $X \times X$. We say that $x, y \in X$ are *related* if $(x,y) \in \mathcal{R}$ and we write $x\mathcal{R}y$.

An *equivalence relation* $\mathcal{R}$ on X is a relation that is

- **reflexive**: $x\mathcal{R}x$ for every $x \in X$;
- **symmetric**: $x\mathcal{R}y \Rightarrow y\mathcal{R}x$;
- **transitive**: if $x\mathcal{R}y$ and $y\mathcal{R}z$ then $x\mathcal{R}z$.

The *equivalence class of an element* $x \in X$ is the set

$$[x] = \{y \in X : y\mathcal{R}x\}.$$

The set of equivalence classes is denoted by $X/\mathcal{R}$, and it is called the *quotient space* of X with respect to $\mathcal{R}$.

The equivalence classes form a partition of X.

Recall that a *partition* of a non-empty set X is a collection $\{U_i : i \in I\}$ of subsets of X such that

- $U_i \neq \emptyset$ for every $i \in I$;
- $U_i \cap U_j = \emptyset$ for every $i \neq j$, $i, j \in I$.
- $\bigcup_{i \in I} U_i = X$.

Remark 3.2. *Defining an equivalence relation on a set X is **equivalent to defining a partition of** X.*

Indeed, given an equivalence relation on a set X, the equivalence classes compose a partition of X.

Conversely, given a partition $\{U_i : i \in I\}$ of X, one can define an equivalence relation $\mathcal{R}$ by

$$x\mathcal{R}y \Leftrightarrow \exists\, i \in I \ \text{ such that } \ \{x, y\} \subseteq U_i\,.$$

Let $p\colon X \to X/\mathcal{R}$ be the map that assigns to each point x in X the equivalence class $[x]$. We call this the *collapsing map.* (This is not standard terminology.)

Proposition 3.3. *Let $(X, \mathcal{T})$ be a topological space, and $\mathcal{R}$ an equivalence relation on X.*

The family $\tilde{\mathcal{T}}$ of subsets $\tilde{U}$ in $X/\mathcal{R}$ such that $p^{-1}(\tilde{U}) \in \mathcal{T}$ is a topology for $X/\mathcal{R}$, called the quotient topology.

Proof. (T1) $p^{-1}(\emptyset) = \emptyset$ and $p^{-1}(X/\mathcal{R}) = X$ are both in $\mathcal{T}$.

(T2) and (T3) follow from the fact that

$$p^{-1}\left(\tilde{U} \cap \tilde{V}\right) = p^{-1}\left(\tilde{U}\right) \cap p^{-1}\left(\tilde{V}\right)$$

and that

$$p^{-1}\left(\bigcup_{i \in I} \tilde{U}_i\right) = \bigcup_{i \in I} p^{-1}(\tilde{U}_i)\,.$$

□

Remark 3.4. Each equivalence class in X becomes a single point in $X/\mathcal{R}$.

Remark 3.5. It follows from the definition of the quotient topology that the collapsing map $p\colon X \to X/\mathcal{R}$ is continuous.

Definition 3.6. We say that two points x, y in X are *identified* if $x\mathcal{R}y$.

Proposition 3.7. *A subset $\tilde{V}$ in $X/\mathcal{R}$ endowed with the quotient topology is closed if and only if $p^{-1}(\tilde{V})$ is closed in X.*

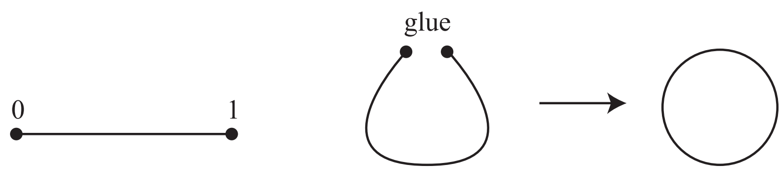
Proof. This follows from the fact that

$$p^{-1}\left((X/\mathcal{R}) \setminus \tilde{V}\right) = X \setminus p^{-1}\left(\tilde{V}\right)\,.$$

□

Example 3.8. Let $X = [0, 1]$, with the standard topology. Define the equivalence relation $\mathcal{R}$ on X by the following equivalence classes:

- for each $x \in (0, 1)$, the singleton $\{x\}$ is an equivalence class;
- $\{0, 1\}$ is an equivalence class.



So, the endpoints of the interval $[0,1]$ have been identified. Intuitively, it seems reasonable that the resulting quotient space is homeomorphic to the circle S^1, as suggested below.

This is indeed the case. A formal proof uses the following result.

Proposition 3.9. *Let X and Z be topological spaces. Let $\mathcal{R}$ be an equivalence relation on X. Let $g\colon X/\mathcal{R} \to Z$ be a function. Then g is continuous if and only if $g \circ p\colon X \to Z$ is continuous.*

Proof. ($\Rightarrow$) Suppose that g is continuous. By Remark 3.5 p is continuous. So, $g \circ p$ is continuous.

($\Leftarrow$) Suppose that $g \circ p$ is continuous. Let U be an open subset of Z. Then, by assumption, $(g \circ p)^{-1}(U)$ is open in X. This is $p^{-1}g^{-1}(U)$. By the definition of the quotient topology, $g^{-1}(U)$ is therefore open in $X/\mathcal{R}$. So, g is continuous. $\square$

Example 3.10. Let us return to Example 3.8. Define

$$\begin{aligned} [0,1] &\xrightarrow{f} S^1(\subset \mathbb{R}^2) \\ t &\mapsto (\cos 2\pi t, \sin 2\pi t). \end{aligned}$$

Since $f(0) = f(1)$, this gives a well-defined function

$$\begin{aligned} [0,1]/\mathcal{R} &\xrightarrow{g} S^1 \\ [t] &\mapsto f(t). \end{aligned}$$

So, $f = g \circ p$. Therefore, g is continuous by Proposition 3.9. Now, $[0,1]$ is compact and hence so is $p([0,1]) = [0,1]/\mathcal{R}$. Also, g is a bijection. Finally, S^1 is Hausdorff. So, by Theorem 2.25, g is a homeomorphism. Therefore, $[0,1]/\mathcal{R}$ and S^1 are homeomorphic, as expected.

This style of argument can be generalised, as in the following proposition.

Proposition 3.11. *Let $f\colon X \to Y$ be a surjective continuous map between topological spaces. Let $\mathcal{R}$ be the equivalence relation on X defined by the partition $\{f^{-1}(y) : y \in Y\}$. If X is compact and Y is Hausdorff, then $X/\mathcal{R}$ and Y are homeomorphic.*

Proof. Define $g\colon X/\mathcal{R} \to Y$ by $[x] \mapsto f(x)$. This is well-defined by the definition of $\mathcal{R}$. Now, $g \circ p = f$ where $p\colon X \to X/\mathcal{R}$ is $x \mapsto [x]$. So, Proposition 3.9, g is continuous. It is injective by the definition of $\mathcal{R}$, and it is surjective because f is. Since X is compact, so is $X/\mathcal{R}$, by Proposition 2.19. So, $g\colon X/\mathcal{R} \to Y$ is a continuous bijection from a compact space to a Hausdorff space. By Theorem 2.25, g is a homeomorphism. $\square$

Examples 3.12. (1) Let $X = [0,1] \times [0,1]$, and let $\mathcal{R}$ be the equivalence relation specified by the following equivalence classes:

- $\{(x,0),(x,1)\}$, with x arbitrary number in the interval $(0,1)$,
- $\{(0,y),(1,y)\}$, with y arbitrary in the interval $(0,1)$,
- $\{(0,0),(1,0),(0,1),(1,1)\}$
- and $\{(x,y)\}$ with x,y two arbitrary numbers in the interval $(0,1)$.

The quotient space $([0,1]\times[0,1])/\mathcal{R}$ is *homeomorphic to the* 2*-dimensional torus* $\mathbb{T}^2$.

Let $f\colon [0,1]\times[0,1]\to\mathbb{T}^2$ be $(t_1,t_2)\mapsto\left(e^{2i\pi t_1},e^{2i\pi t_2}\right)$. The partition $\{f^{-1}(y) : y\in\mathbb{T}^2\}$ is precisely $\mathcal{R}$. Hence, by Proposition 3.11, $([0,1]\times[0,1])/\mathcal{R}$ is homeomorphic to $\mathbb{T}^2$.

(2) Consider the closed planar unit disk

$$\mathbb{D}^2 = \{(x,y)\in\mathbb{R}^2 : x^2+y^2\leq 1\}.$$

Let $\mathcal{R}$ be the equivalence relation specified by the following equivalence classes:

- all the singletons in the open disk $\{(x,y)\}$ with $x^2+y^2<1$
- and the boundary circle $\mathbb{S}^1$.

The quotient space $\mathbb{D}^2/\mathcal{R}$ is homeomorphic to the 2-dimensional sphere

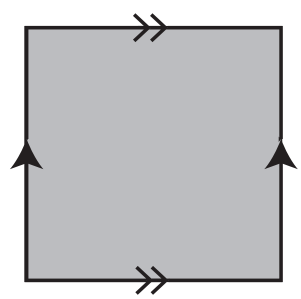
$$\mathbb{S}^2 = \{(x,y,z) : x^2+y^2+z^2=1\}.$$

This is proved by applying Proposition 3.11 to the map $f:\mathbb{D}^2\to\mathbb{S}^2$, $f(0,0)=(0,0,1)$, and for $(x,y)\neq(0,0)$

$$f(x,y)=\left(\frac{x}{\sqrt{x^2+y^2}}\sin\left(\pi\sqrt{x^2+y^2}\right),\ \frac{y}{\sqrt{x^2+y^2}}\sin\left(\pi\sqrt{x^2+y^2}\right),\ \cos\left(\pi\sqrt{x^2+y^2}\right)\right).$$

In the case of the torus above, there is a convenient way of specifying the equivalence relation, by means of a picture.

One starts with the square, and one draws a single arrow on each of the left and right edges. This means 'identify each point on the left edge with the corresponding point on the right edge'. Similarly, we draw double arrows on the top and bottom edges.



Of course, not every equivalence relation can be specified in this way. For example, the second of the above examples, where all the boundary of a disc is identified, cannot be so expressed.

It's important to realise that in the above type of description of a quotient space, *there might be points that are identified in a way that is not immediately obvious.* For example, the top-left corner is identified with the top-right corner because they are both endpoints of single arrows. And the top-right corner is identified with the bottom-right corner, because they are both endpoints of double arrows. Hence, the top-left corner is identified with the bottom-right corner, because an equivalence relation is transitive.

More formally, we are using the following construction.

Definition 3.13. Let X be a set and let $\mathcal{R}$ be a relation on X that is reflexive and symmetric, but not necessarily transitive. Then, the *transitive closure* of $\mathcal{R}$ is the relation $\overline{\mathcal{R}}$ given by

$$x\overline{\mathcal{R}}x' \Leftrightarrow \begin{array}{c}\text{there is a finite sequence of points } x_1,\dots,x_n \text{ in } X \text{ such that}\\ x=x_1\mathcal{R}x_2\mathcal{R}\dots\mathcal{R}x_{n-1}\mathcal{R}x_n=x'.\end{array}$$

The following is immediate.

Lemma 3.14. *The transitive closure of a symmetric reflexive relation is an equivalence relation.*

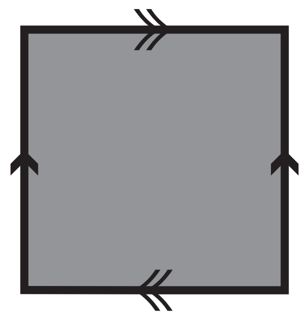
Proof. (Reflexive) For $x \in X$, set $x_1 = x_2 = x$. Then $x = x_1 \mathcal{R} x_2 = x$, and so $x \overline{\mathcal{R}} x$.

(Symmetric) If $x \overline{\mathcal{R}} x'$, then $x = x_1 \mathcal{R} x_2 \mathcal{R} \ldots \mathcal{R} x_{n-1} \mathcal{R} x_n = x'$. So, $x' = x_n \mathcal{R} x_{n-1} \mathcal{R} \ldots \mathcal{R} x_2 \mathcal{R} x_1 = x$, and so $x' \overline{\mathcal{R}} x$.

(Transitive) If $x \overline{\mathcal{R}} y$, then $x = x_1 \mathcal{R} x_2 \mathcal{R} \ldots \mathcal{R} x_{n-1} \mathcal{R} x_n = y$. If $y \overline{\mathcal{R}} z$, then $y = y_1 \mathcal{R} y_2 \mathcal{R} \ldots \mathcal{R} y_{n-1} \mathcal{R} y_n = z$. So, $x = x_1 \mathcal{R} x_2 \mathcal{R} \ldots \mathcal{R} x_{n-1} \mathcal{R} x_n \mathcal{R} y_2 \mathcal{R} \ldots \mathcal{R} y_{n-1} \mathcal{R} y_n = z$. Therefore $x \overline{\mathcal{R}} z$. □

We will often specify quotient spaces by first giving a symmetric reflexive relation on a space, and then using its transitive closure. In fact, we will do this so frequently that we will often not say explicitly that this is what we are doing.

Example 3.15. Another famous example is the *Klein bottle*. This is obtained from the square $[0, 1] \times [0, 1]$ by making the following identifications:



Note that, again, the four corners of the square are all identified.

3.2. **Separation axioms.** Unfortunately, it is possible for quotient spaces to have quite nasty topologies, even when the original space was well-behaved.

Example 3.16. Let X be $\mathbb{R}$ with its standard topology. Define $\mathcal{R}$ on X by

$$x_1 \mathcal{R} x_2 \Leftrightarrow x_1 - x_2 \in \mathbb{Q}.$$

We denote the quotient space by $\mathbb{R}/\mathbb{Q}$.

We will show that *the quotient topology on* $\mathbb{R}/\mathbb{Q}$ *is the indiscrete topology.*

Let U be an open non-empty set in $\mathbb{R}/\mathbb{Q}$. Then $p^{-1}(U)$ is open and non-empty in $\mathbb{R}$.

For every $x \in \mathbb{R}$, the set $-x + p^{-1}(U)$ is open and non-empty in $\mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $q \in (-x + p^{-1}(U)) \cap \mathbb{Q}$.

So, $x + q \in p^{-1}(U)$. The set $p^{-1}(U)$ contains any equivalence class that it intersects, therefore $x + \mathbb{Q} \subseteq p^{-1}(U) \Leftrightarrow p(x) \in U$.

We have thus proved that for every $x \in \mathbb{R}$, $p(x) \in U$. It follows that $U = \mathbb{R}/\mathbb{Q}$.

Thus the quotient topology on $\mathbb{R}/\mathbb{Q}$ is the indiscrete topology.

Therefore, a quotient space $X/\mathcal{R}$ can be non-Hausdorff even when X is Hausdorff.

We now embark on a more systematic analysis of separation axioms for a quotient space. These axioms are phased using open subsets of a quotient space. These in turn can be understood as special types of subsets of the original space, as follows.

Definition 3.17. Let X be a topological space, and let $\mathcal{R}$ be an equivalence relation on X. Then a subset A of X is *saturated* (with respect to $\mathcal{R}$) if it is a union of equivalence classes.

Lemma 3.18. *Let $\mathcal{R}$ be an equivalence relation on a space X, let $p : X \to X/\mathcal{R}$ be the collapsing map $x \mapsto [x]$ and let A be a subset in X. The following are equivalent:*

(1) *A is saturated;*
(2) *if A intersects an equivalence class $[x]$ (i.e. $A \cap [x] \neq \emptyset$) then A contains $[x]$;*
(3) *whenever $x \in A$ and $y\mathcal{R}x$ the element y is also in A;*
(4) *$A = p^{-1}(p(A))$.*

The proof of the equivalence is easy and left as an exercise.

Lemma 3.19. *There are the following one-one correspondences:*

$$\text{subsets of } X/\mathcal{R} \longleftrightarrow \text{saturated subsets of } X$$
$$\text{open subsets of } X/\mathcal{R} \longleftrightarrow \text{open saturated subsets of } X$$
$$\text{closed subsets of } X/\mathcal{R} \longleftrightarrow \text{closed saturated subsets of } X$$

Proof. Associated with each subset B of $X/\mathcal{R}$, its inverse image $p^{-1}(B)$ is a saturated subset of X. Associated with a saturated subset A of X, its image $p(A)$ is the corresponding subset of $X/\mathcal{R}$. It is easy to see that this sets up a one-one correspondence between subsets of $X/\mathcal{R}$ and saturated subsets of X. (Note that $p(p^{-1}(B)) = B$ for any surjective map p, and $p^{-1}(p(A)) = A$ when A is saturated, by Lemma 3.18.)

Note that $A \subseteq X/\mathcal{R}$ is open if and only if $p^{-1}(A)$ is open in X, by the definition of the quotient topology. Similarly $A \subseteq X/\mathcal{R}$ is closed if and only if $p^{-1}(A)$ is closed in X. This gives the second and third correspondences. ☐

Proposition 3.20. *Let X be a topological space, and let $X/\mathcal{R}$ be a quotient space. Then $X/\mathcal{R}$ satisfies the first separation axiom if and only if every equivalence class in X is closed.*

Proof. The first separation axiom is equivalent to the statement that every singleton in $X/\mathcal{R}$ is closed.

Under the correspondence in Lemma 3.19, each singleton in $X/\mathcal{R}$ corresponds to an equivalence class in X. ☐

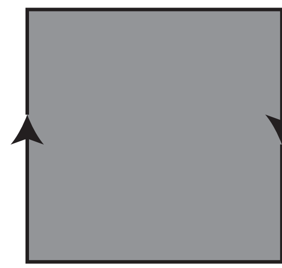
Another application of 3.19 immediately gives the following.

Proposition 3.21. *A quotient topological space is Hausdorff if and only if any two distinct equivalence classes are contained in two disjoint open saturated sets.*

Example 3.22. Let $X = [0,1] \times [0,1]$, and let $\mathcal{R}$ be the equivalence relation with the following equivalence classes:

- the singleton $\{(x,y)\}$ for each $x \in (0,1)$ and $y \in [0,1]$,
- the pair $\{(0,y),(1,1-y)\}$ for each $y \in [0,1]$.

Pictorially, $X/\mathcal{R}$ is as shown. It is the famous *Möbius band*.



We claim that it is Hausdorff. To prove this using Proposition 3.21, we need to consider any two distinct equivalence classes. There are a number of possibilities for these equivalence classes. Let us focus just on the case where one is a pair $\{(0,y),(1,1-y)\}$ and one is a singleton $\{(a,b)\}$ with $a \in (0,1)$.

For $\varepsilon > 0$ and $\delta > 0$ small enough, the open saturated sets

$$[B((0,y),\varepsilon) \cup B((1,1-y),\varepsilon)] \cap X \text{ and } B((a,b),\delta) \subseteq X$$

are disjoint.

Similar arguments work for any two distinct equivalent classes in $X/\mathcal{R}$.

Examples 3.23. Let $\mathcal{R}$ be the equivalence relation on $\mathbb{R}$ given by $x\mathcal{R}y \Leftrightarrow x-y \in \mathbb{Z}$. The quotient space $\mathbb{R}/\mathcal{R}$ is usually denoted $\mathbb{R}/\mathbb{Z}$.

$\mathbb{R}/\mathbb{Z}$ *is Hausdorff.*

Let $[x] \neq [y] \in \mathbb{R}/\mathcal{R}$. Let $m = \min\{|x-y+n| : n \in \mathbb{Z}\}$ and $\epsilon = m/2$.

Then the sets:

$$U_1 = \bigcup_{n\in\mathbb{Z}} (x-\epsilon+n, x+\epsilon+n), \quad U_2 = \bigcup_{n\in\mathbb{Z}} (y-\epsilon+n, y+\epsilon+n)$$

are disjoint open saturated sets such that $[x] \subseteq U_1, [y] \subseteq U_2$. It follows that $p(U_1), p(U_2)$ are open disjoint sets containing $[x],[y]$ respectively, so $\mathbb{R}/\mathbb{Z}$ is Hausdorff.

A similar argument shows that if we impose the equivalence relation on $\mathbb{R}^n$ by $x\mathcal{R}y \Leftrightarrow x-y \in \mathbb{Z}^n$, then the resulting quotient space is Hausdorff.

Definition 3.24. Real n-dimensional projective space $\mathbb{R}P^n$ is the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ with respect to the equivalence relation $\mathcal{R}$ where $\mathbf{x}\mathcal{R}\mathbf{y}$ if and only if there exists $\lambda \neq 0$ such that $\mathbf{x} = \lambda \mathbf{y}$.

In other words $\mathbb{R}P^n$ is the set of lines in $\mathbb{R}^{n+1}$ through 0.

Example 3.25. For any $n \geq 1$, $\mathbb{R}P^n$ is Hausdorff.

Suppose that l_1, l_2 are distinct lines in $\mathbb{R}^{n+1}$. Let x_1, x_2 be the two points on l_1 with $||x_1|| = ||x_2|| = 1$ and let y_1, y_2 be the two points on l_2 with $||y_1|| = ||y_2|| = 1$.

Let $m = \min\{||x_i - y_j|| : 1 \leq i,j \leq 2\}$. If $\epsilon = m/2$ we set

$$U_1 = \{l \text{ line through } 0 \text{ in } \mathbb{R}^n : l \cap B(x_1,\epsilon) \neq \emptyset\}$$

,

$$U_2 = \{l \text{ line through } 0 \text{ in } \mathbb{R}^n : l \cap B(y_1,\epsilon) \neq \emptyset\}.$$

Note that if $l \in U_1 \cap U_2$ and $z \in l$ is such that $||z|| = 1$ then $||z - x_i|| < \epsilon$, $||z - y_j|| < \epsilon$ for some $i, j \in \{1, 2\}$. However $||y_j - x_i|| \geq 2\epsilon$ so this is impossible and the sets U_1, U_2 are disjoint.

They are also open and saturated as subsets of $\mathbb{R}^n$, and $l_1 \in U_1, l_2 \in U_2$. It follows that $\mathbb{R}P^n$ is Hausdorff.

For the next proposition we use the following notation:

(a) The *unit sphere in* $\mathbb{R}^3$ is $\mathbb{S}^2 = \{(x,\, y,\, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$;

(b) The *upper hemisphere in* $\mathbb{S}^2$ is $D^+ = \{(x,\, y,\, z) \in \mathbb{S}^2 : z \geqslant 0\}$;

(c) The *closed unit disc in* $\mathbb{R}^2$ is $\mathbb{D}^2 = \{(x,\, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\}$.

Proposition 3.26. *The following are all* ***homeomorphic to the projective plane*** $\mathbb{R}P^2$.

(a) *The quotient space* $\mathbb{S}^2/\mathcal{R}$ *where* $\mathcal{R}$ *identifies each pair of antipodal points of* $\mathbb{S}^2$.

(b) *The quotient space* $D^+/\mathcal{R}$ *where* $\mathcal{R}$ *identifies each pair of antipodal points on the boundary circle of* D^+.

(c) *The quotient space* $\mathbb{D}^2/\mathcal{R}$ *where* $\mathcal{R}$ *identifies each pair of antipodal points on the boundary circle of* $\mathbb{D}^2$.

(d) *The quotient space of the square* $[0,\, 1] \times [0,\, 1]$ *by the equivalence relation which identifies* $(s,\, 0),\ (1 - s,\, 1)$ *for each* $s \in [0,\, 1]$ *and* $(0,\, t),\ (1,\, 1 - t)$ *for each* $t \in [0,\, 1]$.

Remarks 3.27. (1) Proposition 3.26 implies that *the projective plane is compact.*

(2) A proposition similar to Proposition 3.26 can be formulated for every projective space $\mathbb{R}P^n$, $n \geq 2$.

Proof. To prove (a) and (b) it suffices to restrict the collapsing map $p : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}P^2$ to $\mathbb{S}^2$, respectively D^+, and apply Proposition 3.11

(c) The stereographic projection $\pi : \mathbb{R}^2 \to \mathbb{S}^2$,

$$\pi(x) = \left(\frac{2x}{1 + x^2 + y^2} \,,\, \frac{2y}{1 + x^2 + y^2} \,,\, \frac{1 - x^2 - y^2}{1 + x^2 + y^2} \right) ,$$

restricted to the unit disk $\mathbb{D}^2$ defines a homeomorphism between $\mathbb{D}^2$ and the upper hemisphere.

The equivalence relation in (b) becomes the equivalent relation in (c). So, $D^+/\mathcal{R}$ is also homeomorphic to $\mathbb{R}P^2$.

(d) follows from the fact that the unit disk $\mathbb{D}^2$ and the unit square $[0, 1] \times [0, 1]$ are homeomorphic. □

3.3. **Quotient maps.**

Proposition 3.28 (a second way of defining the quotient topology)**.** *Let* $p \colon X \to X/\mathcal{R}$ *be* $x \mapsto [x]$. *Then the quotient topology on* $X/\mathcal{R}$ *is the finest topology on* $X/\mathcal{R}$ *making* p *continuous.*

Proof. Let $\tilde{\mathcal{T}}'$ be a topology on $X/\mathcal{R}$ such that $p : X \to \left(X/\mathcal{R}\,,\, \tilde{\mathcal{T}}'\right)$ is continuous. Then for every $U \in \tilde{\mathcal{T}}'$, $p^{-1}(U)$ is in $\mathcal{T}$.

It follows that $U \in \tilde{\mathcal{T}}$. We have thus proved that $\tilde{\mathcal{T}}' \subseteq \tilde{\mathcal{T}}$. □

Definition 3.29. A function $f\colon X \to Y$ between topological spaces is an *open mapping* if, for each open set U in X, $f(U)$ is open in Y.

Remarks 3.30. (1) Obviously, not every continuous map is an open map.

(2) If $\mathcal{R}$ is an equivalence relation on a topological space X, and $p\colon X \to X/\mathcal{R}$ is the map $x \mapsto [x]$, then p *need not be an open mapping*!

Examples 3.31. (1) The map

$$\begin{aligned} \mathbb{R} &\xrightarrow{f} S^1 \quad (\subset \mathbb{C}) \\ t &\mapsto e^{2\pi i t}. \end{aligned}$$

is an open mapping.

(2) Consider Example 3.22 and the map $p\colon [0,1] \times [0,1] \to ([0,1] \times [0,1])/\mathcal{R}$ sending the square to the Möbius band M. This is not an open mapping.

For example, let U be the ball of radius $\frac{1}{2}$ about $(0,0)$ in $[0,1] \times [0,1]$. This is open subset of $[0,1] \times [0,1]$. But its image $p(U)$ is not open in M. To prove this, note that $p(U)$ is open in M if and only if $p^{-1}(p(U))$ is open in $[0,1] \times [0,1]$. The set $p^{-1}(p(U))$ equals

$$U \cup \{(1,y) : 1/2 < y \leq 1\},$$

which is not open in $[0,1] \times [0,1]$.

We now define the notion of a quotient map, which is useful ***when trying to prove that a quotient space is homeomorphic to another topological space****.*

Definition 3.32. A map $p\colon X \to Y$ between topological spaces is a *quotient map* if

(1) p is surjective, and
(2) for every $U \subseteq Y$, U is open if and only if $p^{-1}(U)$ is open.

Proposition 3.33. *Let $\mathcal{R}$ be an equivalence relation on a space X, endow $X/\mathcal{R}$ with the quotient topology, and let $p : X \to X/\mathcal{R}$ be the collapsing map $x \mapsto [x]$. Then p is a quotient map.*

Proof. This follows immediately from the definitions of the quotient topology and quotient map. □

Remark 3.34. It follows from the definition that a quotient map is continuous. A surjective continuous open mapping is a quotient map. However, a quotient map need not be an open mapping. (See Example 3.31 (2).)

There is an important relationship between quotient maps and quotient spaces, as explained by the following proposition.

Proposition 3.35. *Suppose that $q : X \to Y$ is a quotient map and that $\mathcal{R}$ is the equivalence relation on X corresponding to the partition $\{q^{-1}(y) : y \in Y\}$. Then $X/\mathcal{R}$ and Y are homeomorphic.*

Proof. Consider the map $g\colon X/\mathcal{R} \to Y$ given by $g([x]) = q(x)$. It is surjective because q is surjective, and it is injective by construction. So, $g\colon X/\mathcal{R} \to Y$ is a bijection. We will show that it is a homeomorphism.

If $p : X \to X/\mathcal{R}$ is the standard collapsing map, then by the definition of g, we have that $g \circ p = q$. In particular, as p is continuous, g is continuous according to Proposition 3.9.

Let U be an open set in $X/\mathcal{R}$. We need to show that $g(U)$ is open, and hence that g^{-1} is continuous. Now, $q^{-1}(g(U)) = p^{-1}(U)$. So, by the definition of a quotient map, $g(U)$ is open if and only if $p^{-1}(U)$ is open. But, $p^{-1}(U)$ is open since p is continuous. □

Examples 3.36. (1) $\mathbb{R}/\mathbb{Z}$ is homeomorphic to the planar unit circle $\mathbb{S}^1$.

Consider the map $f : \mathbb{R} \to \mathbb{S}^1$, $f(t) = e^{2i\pi t} = \cos(2\pi t) + i\sin(2\pi t)$.

Let $\mathcal{R}$ be the equivalence relation $x\mathcal{R}y \Leftrightarrow f(x) = f(y)$. Then $x\mathcal{R}y \Leftrightarrow x - y \in \mathbb{Z}$.

The map f is surjective, continuous and an open mapping.

Proposition 3.35 implies that $\mathbb{R}/\mathbb{Z}$ is homeomorphic to $\mathbb{S}^1$.

(2) $\mathbb{R}^n/\mathbb{Z}^n$ is homeomorphic to $\underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n \text{ times}}$. This space is denoted by $\mathbb{T}^n$ and it is called the n**-dimensional torus**. Note that it is a compact space.

To prove the homeomorphism it suffices to consider the quotient map

$$F(t_1, .., t_n) = \left(e^{2i\pi t_1}, ..., e^{2i\pi t_n}\right)$$

and argue as in (1).

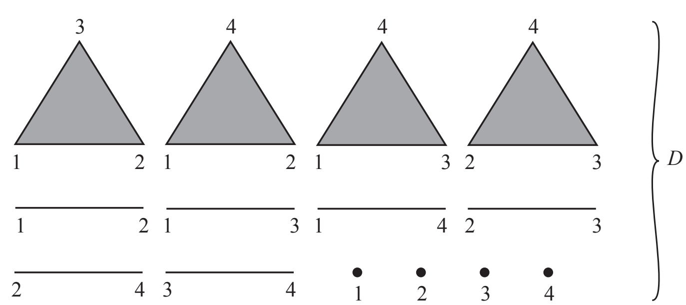
4. Simplicial complexes

4.1. **Definitions.**

Frequently, quite complicated topological spaces can be built out of simple pieces. In this section, we describe one method of doing this. Here, the building blocks are 'simplices', which are like triangles, but generalised to arbitrary dimension. The spaces that we obtain by gluing together simplices in certain ways will be called 'simplicial complexes'.

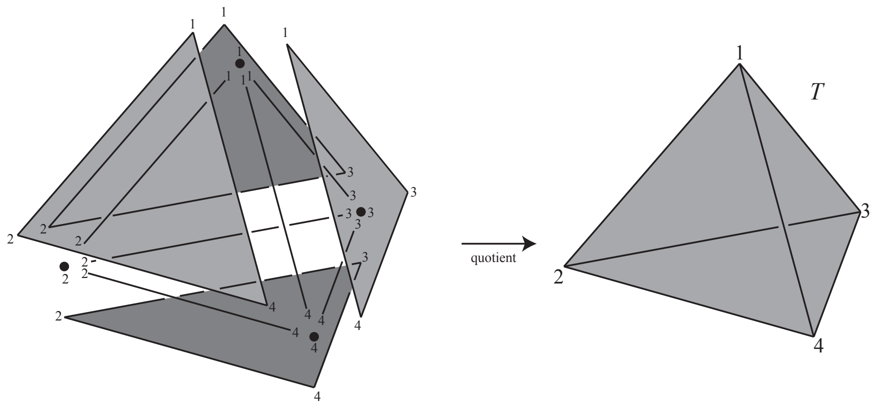
Before we embark on the formal definitions, we give a worked example, which hopefully will be useful.

Example 4.1. Consider a tetrahedron T. This has four vertices, six edges and four triangles. (Note that we are not including the region in $\mathbb{R}^3$ that T bounds.) One way of building T is to start with the disjoint union D of four points, six intervals and four triangles. We give the four points labels 1, 2, 3 and 4. We also label the endpoints of the intervals, and label the corners of the triangles, as shown below.



We now form a quotient space K, obtained by identifying the endpoint of each edge with the corresponding point, and by identifying each of side of each triangle with the corresponding edge. (Here, we are following the convention mentioned after Definition 3.13 that, when we declare that certain points are identified with other points, we are considering the transitive closure of this relation, and then forming the resulting quotient space.)

The resulting quotient space is indeed homeomorphic to the tetrahedron T. To prove this, note that there is an obvious map $f\colon D \to T$ which sends each point, interval or triangle onto the corresponding vertex, edge or face of T. Then two points of D are related by the equivalence relation described above if and only if they have the same image under f. Thus, by Proposition 3.11, there is a homeomorphism from the quotient space K to T.



We want to generalise this construction. The first step will be consider higher-dimensional generalisations of points, intervals and triangles.

Definition 4.2. The *standard n-simplex* is the set

$$\Delta^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i \geq 0 \ \forall i \text{ and } \sum_i x_i = 1\}.$$

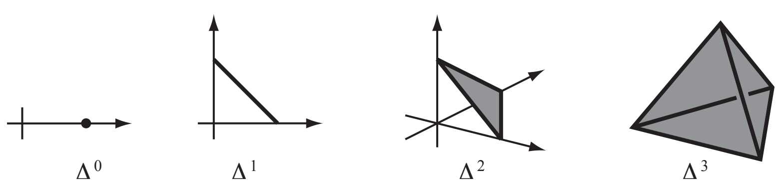
The non-negative integer n is the *dimension* of this simplex. Its *vertices*, denoted $V(\Delta^n)$, are those points $(x_1, \dots, x_{n+1})$ in Δ^n where $x_i = 1$ for some i (and hence $x_j = 0$ for all $j \neq i$). For each non-empty subset A of $\{1, \dots, n+1\}$ there is a corresponding *face* of Δ^n, which is

$$\{(x_1, \dots, x_{n+1}) \in \Delta^n : x_i = 0 \ \forall i \notin A\}.$$

Note that Δ^n is a face of itself (setting $A = \{1, \dots, n+1\}$). The *inside* of Δ^n is

$$\text{inside}(\Delta^n) = \{(x_1, \dots, x_{n+1}) \in \Delta^n : x_i > 0 \ \forall i\}.$$

Note that the inside of Δ^0 is Δ^0.



Remark 4.3. The use of the phrase 'inside of a simplex' is not completely standard. Some authors refer to it as the 'interior' of the simplex or an 'open' simplex. We have chosen not to do this, because 'interior' and 'open' already have other meanings which clash with this usage.

Definition 4.4. A *face inclusion* of a standard m-simplex Δ^m into a standard n-simplex Δ^n (where $m < n$) is a function $\Delta^m \to \Delta^n$ that is the restriction of an injective linear map $\mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ which sends the vertices of Δ^m to vertices of Δ^n.

Note that any injection $V(\Delta^m) \to V(\Delta^n)$ extends to a unique face inclusion. For example, there are six face inclusions $\Delta^1 \to \Delta^2$, corresponding to the six injections $\{1, 2\} \to \{1, 2, 3\}$.

We start with the data used to build our spaces:

Definition 4.5. An *abstract simplicial complex* is a pair (V, Σ), where V is a set (whose elements are called *vertices*) and Σ is a set of non-empty finite subsets of V (called *simplices*) such that

(1) for each $v \in V$, the 1-element set $\{v\}$ is in Σ;
(2) if σ is an element of Σ, so is any non-empty subset of σ.

We say that (V, Σ) is *finite* if V is a finite set.

We now give a method of constructing topological spaces using the above data:

Definition 4.6. The *topological realisation* $|K|$ of an abstract simplicial complex $K = (V, \Sigma)$ is the space obtained by the following procedure:

(1) For each $\sigma \in \Sigma$, take a copy of the standard n-simplex, where $n+1$ is the number of elements of σ. Denote this simplex by Δ_σ. Label its vertices with the elements of σ.
(2) Whenever $\sigma \subset \tau \in \Sigma$, identify Δ_σ with a subset of Δ_τ, via the face inclusion which sends the elements of σ to the corresponding elements of τ.

In other words, $|K|$ is a quotient space, obtained by starting with the disjoint union of the simplices in (1), and then imposing the equivalence relation that is described in (2).

Example 4.7. Let $V = \{1, 2, 3, 4\}$ and let Σ be the following collection of subsets of V:

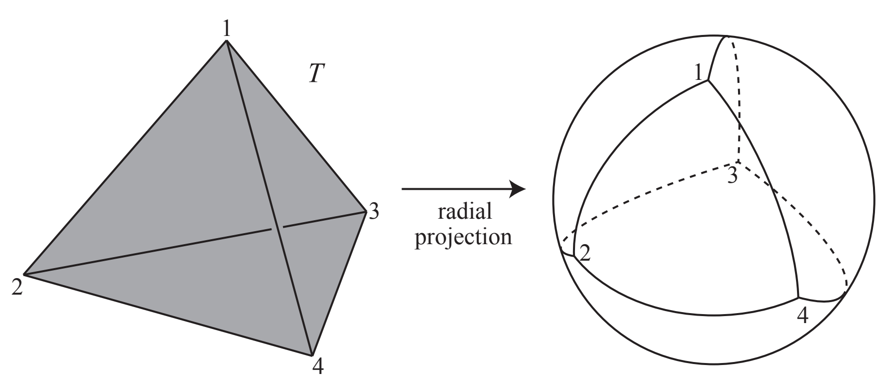
$$\{1,2,3\}, \quad \{1,2,4\}, \quad \{1,3,4\}, \quad \{2,3,4\},$$
$$\{1,2\}, \quad \{1,3\}, \quad \{1,4\}, \quad \{2,3\},$$
$$\{2,4\}, \quad \{3,4\}, \quad \{1\}, \quad \{2\}, \quad \{3\}, \quad \{4\}.$$

Then, to build the topological realisation of (V, Σ), we start with four 0-simplices, six 1-simplices, and four 2-simplices. We identify them exactly as in Example 4.1. So, $|K|$ is the tetrahedron T.

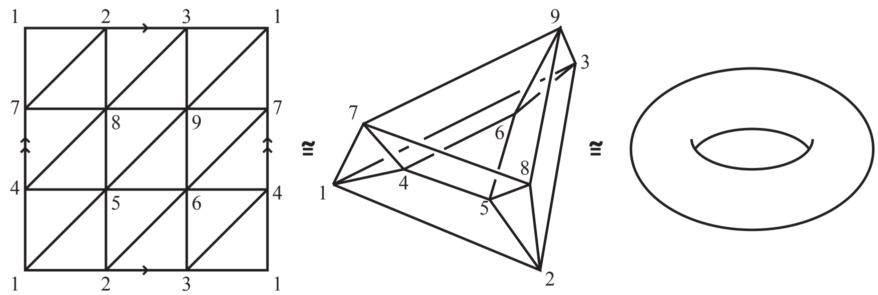
Whenever we refer to a *simplicial complex*, we will mean either an abstract simplicial complex or its topological realisation.

Definition 4.8. A *triangulation* of a space X is a simplicial complex K together with a choice of homeomorphism $|K| \to X$.

Example 4.9. The 2-sphere has a triangulation, as follows. Let K be the simplicial complex given in Example 4.7. We showed there that $|K|$ is homeomorphic to the tetrahedron T. There is clearly a homeomorphism $T \to S^2$. For example, if we arrange T so that it encloses the origin of $\mathbb{R}^3$, then radial projection gives a homeomorphism $T \to S^2$. Hence, we obtain a triangulation of S^2.



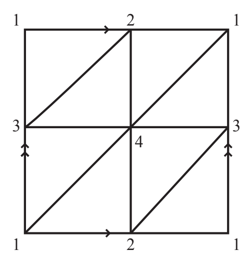
A triangulation of the sphere



A triangulation of the torus

Example 4.10. The torus $S^1 \times S^1$ has a triangulation using nine vertices, as shown below. For reasons of clarity, we have omitted from the middle diagram the edges that are diagonal in the left diagram.

At first sight, this triangulation of the torus may seem be needlessly complicated. Might we have been able to use fewer simplices? For example, one could try to build a triangulation using the following description.



Not a triangulation of the torus

However, this is not a triangulation. It is true that this is a way of building a torus using a collection of simplices. But these simplices do not form a simplicial complex. For example, consider the bottom edge of the square. Here, there are two 1-simplices, both labelled $\{1, 2\}$. But in a simplicial complex, any given set of vertices spans *at most one* simplex. To see this, return to Definition 4.5, and note that Σ is a set of subsets of V. In particular, any given subset of V appears once in Σ or not at all.

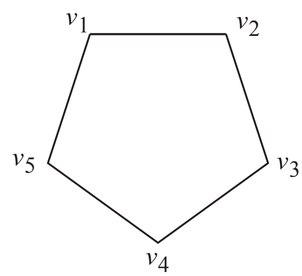
It is clear from the above examples that, when a space admits a triangulation, then it might do so in many different ways. Nevertheless, triangulations are useful structures, as we will see.

Definition 4.11. Let K be a simplicial complex with vertices $\{v_1, \dots, v_n\}$ for some $n \geq 3$, and having just 0-simplices and 1-simplices, where the 1-simplices are precisely $\{v_i, v_{i+1}\}$ for each i between 1 and $n-1$, and $\{v_n, v_1\}$. Then K is a *simplicial circle*.

4.2. **Substructures of a simplicial complex.**

Lemma 4.12. *The topological realisation $|K|$ of a simplicial complex K is the union of the insides of its simplices. Moreover, for distinct simplices, their insides are disjoint.*

Remark 4.13. An alternative way of phrasing this is: *There is a bijection between $|K|$ and the disjoint union of the insides of its simplices.* However, note that these spaces are *not* homeomorphic in general. For example, the disjoint union of two non-empty spaces is always disconnected. On the other hand, many simplicial complexes are connected.



A simplicial circle

Proof. According to Definition 4.6, $|K|$ is obtained from a disjoint union D of simplices, by forming a quotient space. Let $q\colon D \to |K|$ be the collapsing map. It is easy to see that the restriction of q to the insides of D is a bijection. This is because no two distinct points in the insides of simplices are identified, and because every point is identified with a point in the inside of some simplex. The lemma follows immediately. $\square$

Definition 4.14. A *subcomplex* of a simplicial complex $K = (V, \Sigma)$ is a simplicial complex $K' = (V', \Sigma')$, where $V' \subseteq V$ and $\Sigma' \subseteq \Sigma$.

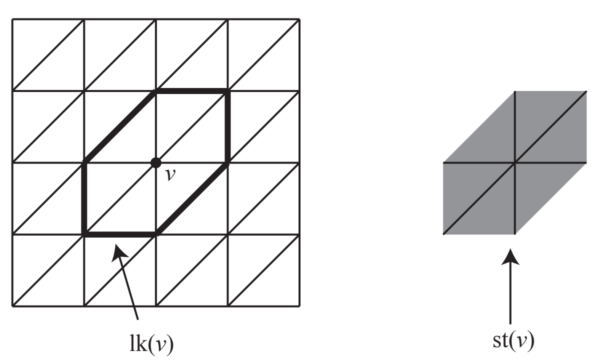
Lemma 4.15. *For a subcomplex K' of a simplicial complex K, its realisation $|K'|$ is a closed subset of $|K|$.*

Proof. The realisation $|K|$ is obtained from a disjoint union of copies of standard simplices, by forming a quotient space. Let D be this disjoint union, and let $q\colon D \to |K|$ be the collapsing map. Then a subset W is closed in $|K|$ if and only if $q^{-1}(W)$ is closed in D, by Proposition 3.7. Let $W = |K'|$. Then $q^{-1}(W)$ is a union of faces in each simplex of D. This is clearly a closed subset of that simplex and hence a closed subset of D. So, $q^{-1}(W)$ is indeed closed. $\square$

Definition 4.16. For a simplicial complex $K = (V, \Sigma)$ and a subset V' of the vertex set V, the subcomplex *spanned by* V' has vertex set V' and consists of all simplices in Σ that have all their vertices in V'.

Definition 4.17. Let $K = (V, \Sigma)$ be a simplicial complex, and let $v \in V$ be a vertex.

(1) The *link* of v, denoted $\mathrm{lk}(v)$, is the subcomplex with vertex set $\{w \in V \backslash \{v\} : \{v, w\} \in \Sigma\}$ and with simplices σ such that $v \notin \sigma$ and $\sigma \cup \{v\} \in \Sigma$.
(2) The *star* of v, denoted $\mathrm{st}(v)$, is $\bigcup\{\mathrm{inside}(\sigma) : \sigma \in \Sigma \text{ and } v \in \sigma\}$.



Lemma 4.18. *The star of a vertex v is an open set containing v.*

Proof. Clearly v lies in $\mathrm{st}(v)$ because $\{v\}$ is a 0-simplex σ and $\mathrm{inside}(\sigma) = \{v\}$. We must show that $\mathrm{st}(v)$ is an open subset of $|K|$. We will show that $|K| \backslash \mathrm{st}(v)$ is a subcomplex of K. Hence, by Lemma 4.15, it is a closed subset of $|K|$, and hence $\mathrm{st}(v)$ is open.

Now, $\mathrm{st}(v)$ is, by definition, $\bigcup \{\mathrm{inside}(\sigma) : \sigma \in \Sigma \text{ and } v \in \sigma\}$. So, by Lemma 4.12, $|K| \backslash \mathrm{st}(v)$ is $\bigcup \{\mathrm{inside}(\sigma) : \sigma \in \Sigma \text{ and } v \notin \sigma\}$. This is precisely the subcomplex spanned by $V \backslash \{v\}$. ☐

4.3. **Elementary properties of simplicial complexes.**

Definition 4.19. An *edge path* in a simplicial complex K is a sequence of vertices $(v_0, \dots, v_n)$ such that for every i, $\{v_i, v_{i+1}\}$ is a simplex of K.

Proposition 4.20. *Let K be a simplicial complex. Then the following are equivalent:*

(1) *Any two vertices in K can be joined by an edge path.*
(2) *$|K|$ is path-connected.*
(3) *$|K|$ is connected.*

Proof. (1) $\Rightarrow$ (2). Suppose that any two vertices in K can be joined by an edge path. Let x and y be any two points in $|K|$. Each lies in a simplex, and so there is a straight line in that simplex joining it to a vertex. These two vertices can be joined by an edge path, which can be realised as a path in $|K|$. So, $|K|$ is path-connected.

(2) $\Rightarrow$ (3). This is Proposition 1.89.

(3) $\Rightarrow$ (1). Suppose that (1) does not hold. Pick any vertex v, and let V' be the collection of all vertices in V that can be joined to v by an edge path. Then, by assumption, V' is a proper subset of V. Let $V'' = V \backslash V'$. Then no edge joins a vertex in V' to one in V''. Let K' and K'' be the subcomplexes spanned by V'. Then $|K'|$ and $|K''|$ are closed subsets of $|K|$ by Lemma 4.15. The fact that $|K|$ and $|K'|$ are disjoint is a consequence of the fact that $V' \cap V'' = \emptyset$. Their union is all of $|K|$. For otherwise, there is a point x in $|K|$ which lies in neither $|K'|$ nor $|K''|$. It lies in some simplex σ. If the vertices of σ lie entirely in V' or entirely in V'', then all the simplex lies in $|K'|$ or $|K''|$, and hence so does x, which is a contradiction. So, some of the vertices of σ lie in V' and some lie in V'', which implies that there is an edge joining a vertex in V' to one in V''. This is a contradiction. Hence, $|K'|$ and $|K''|$ form disjoint closed non-empty subsets of $|K|$, and their union is all of $|K|$. So, $|K|$ is not connected. ☐

Lemma 4.21. *The realisation of any finite simplicial complex is compact.*

Proof. The realisation $|K|$ of a simplicial complex K is obtained from a disjoint union of simplices, by forming a quotient space. The disjoint union of finitely many simplices is compact. Hence, the quotient space is the continuous image of a compact space and therefore compact, by Proposition 2.19. ☐

Proposition 4.22. *For any finite simplicial complex $K = (V, \Sigma)$, there is a continuous injection $f \colon |K| \to \mathbb{R}^n$ for some positive natural number n.*

Proof. Let K' be the simplicial complex with the vertex set V, but where *every* non-empty subset of V is a simplex. Then $|K'|$ is homeomorphic to a standard simplex. The inclusion $|K| \to |K'|$ is a continuous injection. The standard simplex is a subset of $\mathbb{R}^n$. Hence, we obtain the required continuous injection $|K| \to \mathbb{R}^n$. ☐

Corollary 4.23. *The realisation of a finite simplicial complex is metrizable.*

Proof. For a finite simplicial complex K, let $f \colon |K| \to \mathbb{R}^n$ be the continuous injection from Proposition 4.22. This is a homeomorphism onto its image, since $|K|$ is compact and $\mathbb{R}^n$ is Hausdorff. The image is metrizable, since one can use the restriction of the metric on $\mathbb{R}^n$. Hence, $|K|$ is also metrizable. ☐

Corollary 4.24. *The realisation of any finite simplicial complex is Hausdorff.*

Proof. Any metrizable topological space is Hausdorff, by Example 1.49 (a). ☐

5. Surfaces

Definition 5.1. An *n-dimensional manifold* is a Hausdorff topological space M such that every point of M lies in an open set that is homeomorphic to an open set in $\mathbb{R}^n$. We often abbreviate this to an *n-manifold*.

Examples 5.2. (1) $\mathbb{R}^n$ is an n-manifold.

(2) S^n is an n-manifold, since for each $x \in S^n$, $S^n \backslash \{x\}$ is homeomorphic to $\mathbb{R}^n$ by stereographic projection from x. So, for any point $y \in S^n$, pick a point $x \neq y$ and $S^n \backslash \{x\}$ is an open set containing y that is homeomorphic to $\mathbb{R}^n$.

(3) The torus, Klein bottle and n-dimensional projective space $\mathbb{RP}^n$ are all manifolds.

Any open subspace of n-manifold is an n-manifold. These can get quite complicated, and so it is usual to focus attention on manifolds that are compact. These are often called *closed manifolds* (although this terminology is very confusing, because it has little to do with being a closed subset of a space).

Definition 5.3. A 2-manifold is a *surface*.

Examples 5.4. The 2-sphere, torus, Klein bottle and projective plane $\mathbb{RP}^2$ are clearly closed surfaces.

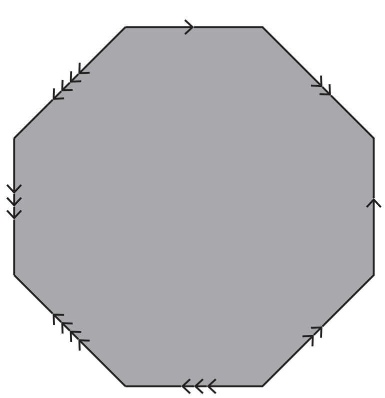
We now give a rather general construction.

5.1. Polygons with a complete set of side identifications.

Let P be a finite-sided convex polygon in $\mathbb{R}^2$. (See the figure below, for example.) Suppose that P has an even number of sides. Arrange these sides into pairs. For each such pair, the two sides of the pair will be identified. More specifically, suppose that e and e' are two sides of a pair. Let e run from (x_0, y_0) to (x_1, y_1), and let e' run from (x'_0, y'_0) to (x'_1, y'_1). Then, as t runs from 0 to 1, the point $(1-t)(x_0, y_0) + t(x_1, y_1)$ lies on e and $(1-t)(x'_0, y'_0) + t(x'_1, y'_1)$ lies on e'. For each $t \in [0, 1]$, we identify these two points.

Once one has chosen the edges e and e' to identify, there is still some choice about how this identification is made, because we can choose e to run from (x_0, y_0) to (x_1, y_1), or the other way round. We encode this choice by drawing an arrow on e, running from (x_0, y_0) to (x_1, y_1). When arrows have been drawn on both e and e', this determines how they are identified.

This construction is a *polygon with a complete set of side identifications.*



An octagon with side identifications

The two-holed torus

Example 5.5. (1) We have already seen three surfaces surface built in this way: the torus, the Klein bottle and $\mathbb{R}P^2$.

(2) We will prove later that the polygon with side identifications shown in the figure above is homeomorphic to the surface also shown there. This surface has several names: the *genus 2 surface* or the *two-holed torus*.

Proposition 5.6. *Any polygon with a complete set of side identifications is a closed surface.*

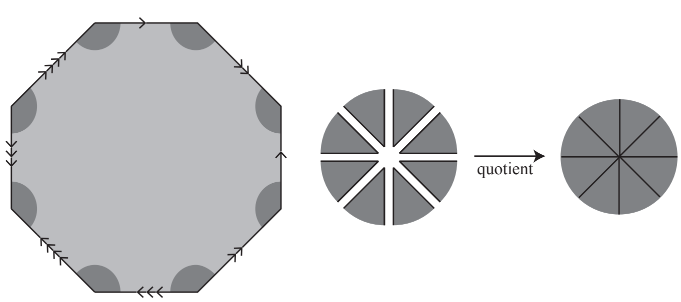
Proof. Let P be the polygon with sides that are to be identified, and let S be the resulting space. Note that there is a homeomorphism taking P to a regular polygon, and so we may assume that P is regular.

We first examine the different types of points in S. We will see that they come in three different flavours. Each point of S is an equivalence class of points in P. We group these equivalence classes into three types:

(1) A point in the inside of P is not identified with any other point, and so forms a singleton equivalence class. We call this an *inside* point of S.
(2) A point lying on an edge of P, but not at a vertex, is identified with exactly one other point of P. This other point also lies on an edge but not at a vertex. These two points form a single point of S which we call an *edge* point.
(3) Several vertices may be identified to form a single point of S. We call such a point a *vertex* point.

For each type of point of S, we now construct an open neighbourhood of the point:

(1) An inside point of S came from a single point x of P. There is an $\epsilon > 0$ such that $B(x, \epsilon)$ lies within the inside of P. So, the subspace $B(x, \epsilon)$ is an open disc. No points of this disc are identified. So, $B(x, \epsilon)$ with the quotient topology is exactly the same as $B(x, \epsilon)$ with its topology as a subspace of P. So, it is an open disc.
(2) An edge point in S comes from two points x and x' in edges of P. In this case, for small enough $\epsilon > 0$, $B(x, \epsilon)$ is homeomorphic to
$$\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < \epsilon^2 \text{ and } x_2 \geq 0\}.$$
Similarly, $B(x', \epsilon)$ is homeomorphic to
$$\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < \epsilon^2 \text{ and } x_2 \leq 0\}.$$
When these are glued, the result is homeomorphic to an open disc in $\mathbb{R}^2$. (Note that one reason that we arranged for P to be regular was so that these two discs patch together correctly.)
(3) A vertex point in S comes from vertices $v_1, \dots, v_n$ in P. Provided $\epsilon > 0$ is small enough, then for each i, $B(v_i, \epsilon)$ is homeomorphic to the following subset of the plane (using polar co-ordinates):
$$\{(r, \theta) : 0 \leq r < \epsilon \text{ and } 0 \leq \theta \leq k\}$$
for some constant k. Also, if ϵ is small enough, these balls in P are disjoint. There is clearly a homeomorphism from this subset of the plane to
$$\{(r, \theta) : 0 \leq r < \epsilon \text{ and } 0 \leq \theta \leq 2\pi/n\},$$
just by rescaling θ. These n pieces of pie glue together to form an open disc in $\mathbb{R}^2$.

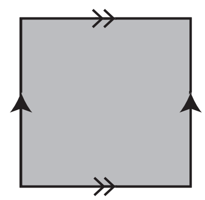


This verifies the main condition in the definition of a surface. But we still need to check that S is Hausdorff. It is clear that, given two distinct points y and y' in S, their open neighbourhoods described above are disjoint, provided ϵ is small enough.

Finally, note that P is compact, and hence the quotient space S is compact. $\square$

5.2. **Two lists of surfaces.** In this subsection, we describe two infinite collections of surfaces, denoted M_g $(g \geq 0)$ and N_h $(h \geq 1)$. Each will be given as a polygon with a complete set of side identifications.

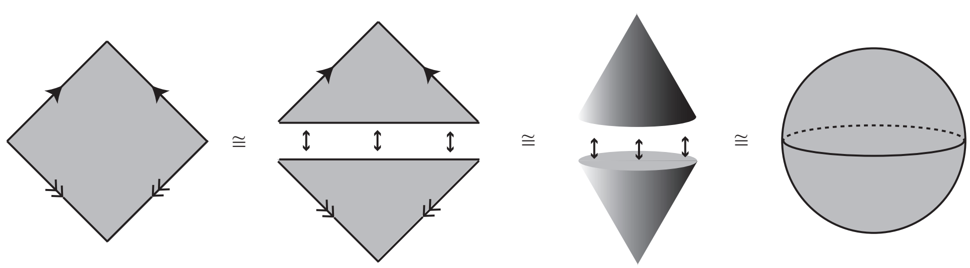
It convenient to describe a polygon with side identifications using a *word*, which is a string of letters, possibly with $(\ )^{-1}$ signs. We start with some vertex of the polygon, and run around the boundary of the polygon. Each pair of edges that is to be identified is given a letter. We orient the edge in some way. When we come to that edge, we write down the letter or its inverse, depending on whether we traverse the edge in the forwards or backwards direction. For example, the standard description of the torus as



gives the word $xyx^{-1}y^{-1}$ when we start at the bottom left corner and run clockwise around the boundary of the square.

We now give the two lists of surfaces. In each list, the first surface is described a little differently from the others.

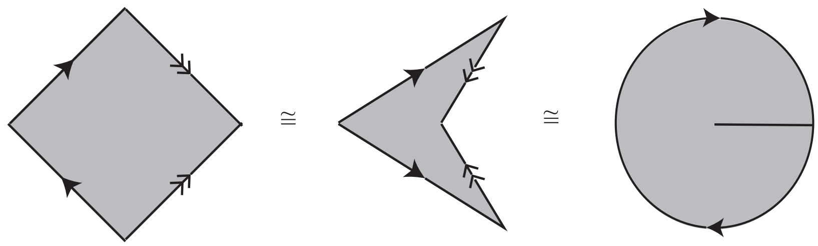
(1) The word $xx^{-1}yy^{-1}$. We denote this surface by M_0. We claim that this is homeomorphic to the 2-sphere. A proof is summarised in the figure below.



In the first step, one divides the square along a diagonal into two triangles. These two triangles will be re-glued later. Each triangle has two of its sides identified, forming a cone (without a base). These two cones are glued along their base circles. The resulting space is clearly homeomorphic to the 2-sphere, via radial projection for example.

(2) For $g \geq 1$, let M_g be the surface obtained from the word $x_1y_1x_1^{-1}y_1^{-1}\dots x_gy_gx_g^{-1}y_g^{-1}$. This is the g*-holed torus*.

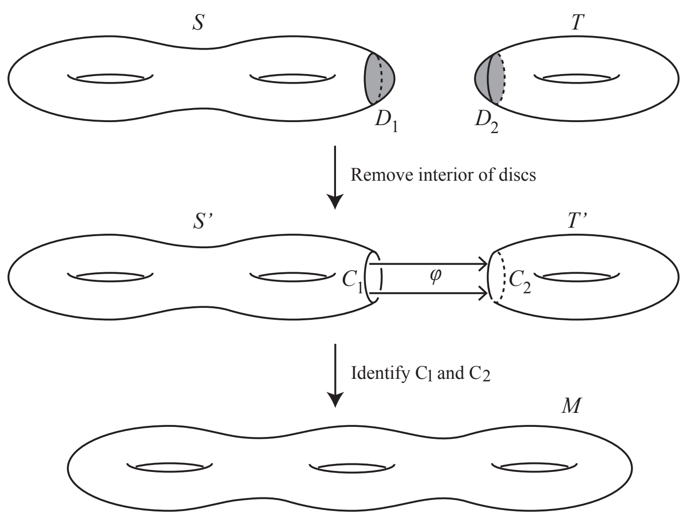
(3) The word $xxyy^{-1}$. We denote this surface by N_1.
This is homeomorphic to the projective plane. A proof is summarised in the figure overleaf. In the first step, we change the shape of a polygon to form a kite. In the second step, the two edges labelled y (with double arrows) are glued, resulting in a disc. The remaining edge labels specify that the top half of the boundary of the disc is glued to the bottom half. In other words, each point z on the boundary is identified with $-z$. This is the projective plane, by Proposition 3.26.



(4) For $h \geq 2$, let N_h be the surface obtained from the word $x_1 x_1 x_2 x_2 \dots x_h x_h$. This is termed the *surface with h crosscaps*.

5.3. **Adding handles and crosscaps.** In the previous subsection, we introduced various surfaces described as polygons with side identifications. In this subsection, we provide a more transparent way of visualising these spaces.

Definition 5.7. Let S be a surface. Let T be a torus. Pick subsets D_1 and D_2 of S and T, each of which is homeomorphic to a closed disc. Remove the interiors of D_1 and D_2 from S and T. Let S' and T' be the resulting spaces. What remains of each disc is its boundary circle. We denote these by C_1 and C_2. Pick a homeomorphism $\phi\colon C_1 \to C_2$. Now form the following topological space: start with the disjoint union of S' and T' and form the quotient space, where each point x on C_1 is identified with $\phi(x)$ on C_2, but no other points are identified. Let M be the resulting space. This is said to be obtained from S by *adding a handle*.



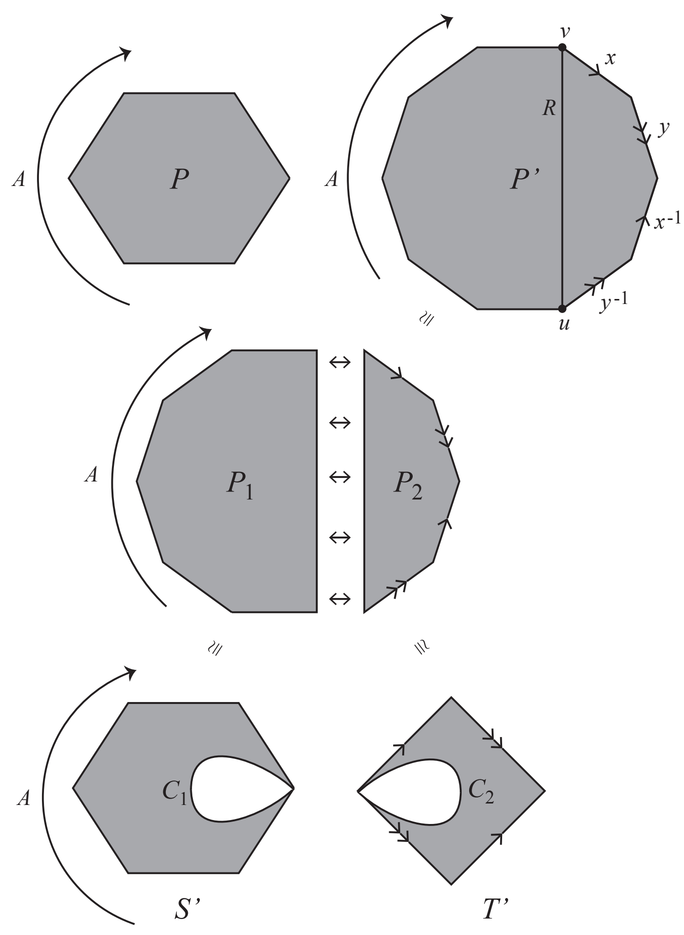
Remarks 5.8. (1) The space M is, in fact, a surface. But, surprisingly, this is not very easy to prove, and we will not do so. The difficulty arises in showing that points coming from C_1 and C_2 have an open neighbourhood homeomorphic to an open subset of $\mathbb{R}^2$.

(2) We made various choices when constructing M. We had to choose the discs D_1 and D_2, and we had to choose the homeomorphism ϕ. It turns out that, if we had made different choices, we would have ended up with a homeomorphic space. Once again, we will not attempt to prove this, and we will not use this fact.

Proposition 5.9. *Let S be obtained from a polygon with a complete set of side identifications. Let A be its boundary word. Suppose that A does not contain the letters x, x^{-1}, y or y^{-1}. Then the word $Axyx^{-1}y^{-1}$ gives a surface that is obtained from S by adding a handle.*

Proof. Let P be the polygon used to build S, and let P' be the polygon with boundary word $Axyx^{-1}y^{-1}$. Let u and v be the points on P' that are at the start and end of the word A. Note that the identifications specified by gluing the edges x to x^{-1} and y to y^{-1} result in gluing u to v. Let R be the straight arc in P' running from u to v. This arc divides P' into two polygons P_1 and P_2.

Consider the polygon P_1. Let us glue its edges together according to the recipe given by A. Let us also identify the two vertices which lie at the endpoints of R. The resulting space S' is homeomorphic to S with the interior of a closed disc removed. In this quotient space, the endpoints of the arc R have been glued up, to form a circle C_1.



The polygon P_2 has five sides. When the four labelled sides are glued together, the result is a torus with the interior of a closed disc removed. Call this space T'. The remains of the arc R form a circle C_2. When the copy of R in P_1 is glued to the copy of R in P_2, this induces a homeomorphism $\phi\colon C_1 \to C_2$. If we glue S' to T' using ϕ, the resulting space is obtained from S by adding a handle. This is the space that is obtained from P' by making the side identifications using the boundary word $Axyx^{-1}y^{-1}$. ☐

Corollary 5.10. *The surface M_g is obtained from a 2-sphere by adding g handles.*

Proof. The surface M_1 is the torus, specified by the word $x_1y_1x_1^{-1}y_1^{-1}$. This is obtained from the sphere by adding one handle. The surface corresponding to the word $x_1y_1x_1^{-1}y_1^{-1}\dots x_gy_gx_g^{-1}y_g^{-1}$ is obtained by adding a further $g-1$ handles, by Proposition 5.9. ☐

Definition 5.11. Let S be a surface. We say that M is obtained from S by *adding a crosscap* if it is obtained as in Definition 5.7, except with the role of the torus T replaced by $\mathbb{RP}^2$.

Thus, one starts with closed discs in M and $\mathbb{RP}^2$, one removes their interiors, and then one identifies their boundary circles via a homeomorphism.

There is an alternative way of understanding this construction. The lemma below states that $\mathbb{RP}^2$ is obtained from a Möbius band and a disc, by identified their boundary circles via a homeomorphism. So, when one removes the interior of this disc from $\mathbb{RP}^2$, the result is a Möbius band. So, adding a crosscap to S is the same as removing the interior of a closed disc from S, and attaching on a Möbius band.

Lemma 5.12. $\mathbb{RP}^2$ *is obtained from a Möbius band and a disc, by identified their boundary circles via a homeomorphism.*

Proof. We saw in Proposition 3.26 that $\mathbb{RP}^2$ is obtained from the unit disc D

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

by identifying each point (x, y) on the boundary of the disc with $(-x, -y)$. Let $\sim$ be this equivalence relation. Let A be the annulus

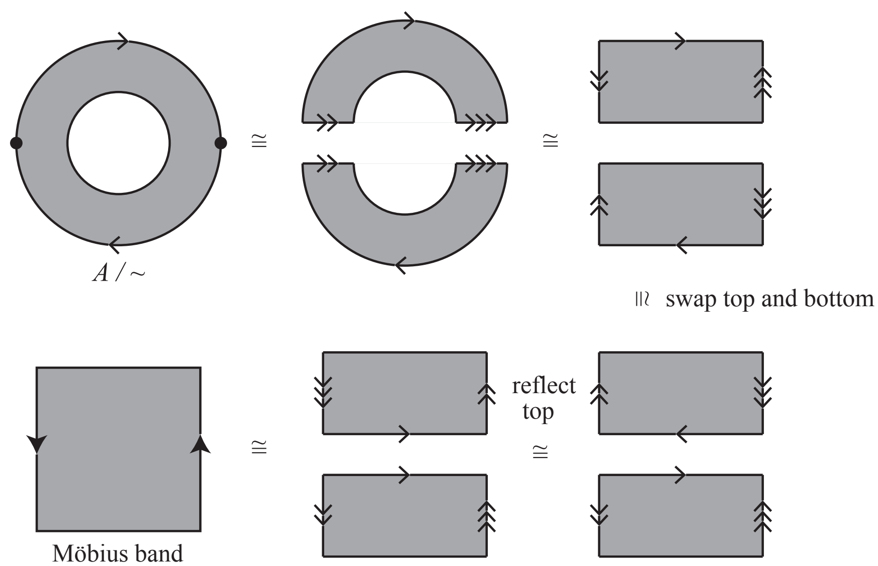
$$\{(x, y) \in \mathbb{R}^2 : 1/4 \leq x^2 + y^2 \leq 1\}$$

and let C be the inner boundary curve

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1/4\}.$$

If one attaches a disc to A along the curve C, the result is D. So, if one attaches a disc to the copy of C in $A/\sim$, the result is $\mathbb{RP}^2$.

Therefore, to prove the lemma, it suffices to show that $A/\sim$ is homeomorphic to the Möbius band. A proof is summarised in the following figure.



$\square$

Proposition 5.13. *Let S be obtained from a polygon with a complete set of side identifications. Let A be its boundary word. Suppose that A does not contain the letters x or x^{-1}. Then the word Axx gives a surface that is obtained from S by adding a crosscap.*

Proof. This is very similar to the proof of Proposition 5.9, and so we only sketch it. We let P be the polygon whose sides are identified to form S. We let P' be the polygon with boundary word Axx. Let u and v be the vertices at the start and end of A, and let R be the straight arc joining them. Cut P' along R, forming two polygons P_1 and P_2. When the sides of P_1 are identified by the recipe given by A, and the vertices u and v are glued together, the result is S with an open disc removed. The triangle P_2 has two of its sides identified. When these are glued, the result is $\mathbb{RP}^2$ with an open disc removed. So, when one reattaches the two curves that came from R, one adds a crosscap to S. ☐

Corollary 5.14. *The surface N_h is obtained from a 2-sphere by adding h crosscaps.*

Proof. We have already seen that N_1, which is $\mathbb{RP}^2$, is obtained from the sphere by adding a crosscap. Recall that the boundary word for N_1 is $xxyy^{-1}$. By Proposition 5.13, the polygon with boundary word $xxyy^{-1}zz$ is obtained from N_1 by adding a crosscap. By Lemma 5.26 (below), $xxzz$ gives the same surface, which is N_2. For $h > 2$, repeatedly apply Proposition 5.13. ☐

5.4. **Closed combinatorial surfaces.** We have already restricted our investigations to closed manifolds. We further simplify the situation by restricting attention to manifolds that admit a triangulation. (In fact, in the case of 2-manifolds, this is no restriction, but that is not easy to prove.) We therefore introduce a simplicial version of Definition 5.1.

Definition 5.15. A *closed combinatorial surface* is connected finite simplicial complex K such that for every vertex v of K, the link of v is a simplicial circle.

Lemma 5.16. *Let K be a closed combinatorial surface. Then*

(1) *every simplex of K has dimension 0, 1 or 2;*
(2) *every 1-simplex of K is a face of exactly two 2-simplices;*
(3) *every point of $|K|$ lies in a 2-simplex;*
(4) *$|K|$ is a surface.*

Proof. (1) Let $\{v_1, \dots, v_{n+1}\}$ be an n-simplex of K. Then the link of v_1 contains $\{v_2, \dots, v_{n+1}\}$. Since this link is a simplicial circle, it consists only of 0-simplices and 1-simplices. So, $n \leq 2$.

(2) Let $\{v_1, v_2\}$ be a 1-simplex. Then $\{v_2\}$ lies in the link of v_1. This link is a simplicial circle. So, there are precisely two 1-simplices in this link that are adjacent to v_2. Hence, there are precisely two 2-simplices in K that have $\{v_1, v_2\}$ as a face.

(3) Each point of $|K|$ lies in the inside of a 2-simplex, a 1-simplex or a 0-simplex. In the first case, there is nothing to prove. In the second case, (2) gives the claim. When the point is a 0-simplex, its link is a simplicial circle and so the vertex lies in a 2-simplex.

(4) Note that $|K|$ is Hausdorff by Corollary 4.24. Each point of $|K|$ lies in the inside of 2-simplex, a 1-simplex or a 0-simplex. In each case, it has a neighbourhood homeomorphic to an open disc in $\mathbb{R}^2$. For a point in the inside of a 2-simplex, this is clear. For a point in the inside of a 1-simplex, this follows from (2). For a vertex, its link is a simplicial circle, and so its star is homeomorphic to an open disc. ☐

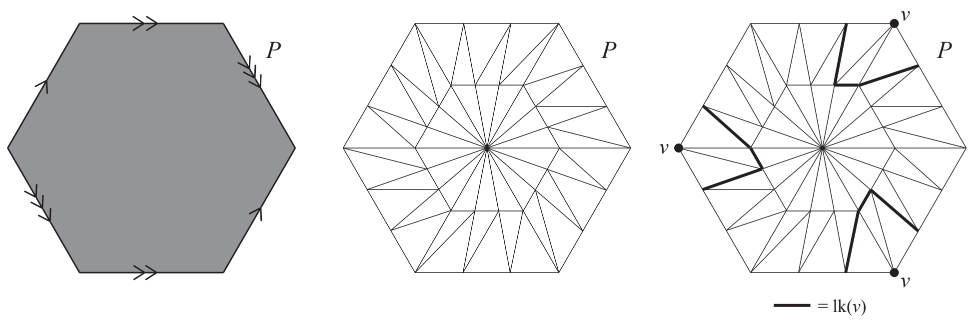
Proposition 5.17. *Any polygon with a complete set of side identifications is homeomorphic to a closed combinatorial surface.*

Proof. Let P be the convex polygon whose sides are identified, and let S be the resulting surface. We need to find a simplicial complex K such that $|K|$ is homeomorphic to S, and so that K satisfies the requirements of a closed combinatorial surface.

We will clearly use the structure of P to construct K. Pick a point O in the interior of P. It is tempting to try the following construction. For each edge e of P, we can consider the triangle with one side e and with opposite vertex O. Each such triangle is a copy of a 2-simplex, and these 2-simplices patch together to form a triangulation for P. However, this does not give a triangulation of S. The reason is that for a given collection of vertices, there may be more than one simplex in S

with these vertices, and this is not allowed in a simplicial complex. More specifically, suppose that e and e' are edges of P that are identified. Then, they become a single 1-simplex after identification. Their endpoints become at most two vertices v and v' (say). Then the triangles attached to e and e' remain distinct 2-simplices after identification, but they have the same vertices O, v and v'.

To get around this problem, we need to start with a finer triangulation of P. One such recipe is shown in the figure below. It is easy to check that this does give a simplicial complex, and that it is a closed combinatorial surface. For example, the three labelled vertices of P shown in the right of the figure become a single vertex v after the sides are identified. The link of v is shown, and it is clearly a simplicial circle. □



5.5. **The classification theorem.** The following result is the main theorem of this chapter.

Theorem 5.18. *Every closed combinatorial surface is homeomorphic to one of the manifolds M_g, for some $g \geq 0$, or N_h, for some $h \geq 1$.*

The proof proceeds in two steps. First we show that any closed combinatorial surface is obtained as a polygon with a complete set of side identifications. Then we show how the polygon and its side identifications can be modified without changing the space up to homeomorphism. By a careful application of these modifications, we show how to change the boundary word until it is in the required standard form.

Proposition 5.19. *If K is a closed combinatorial surface, then $|K|$ is homeomorphic to the space obtained from a $2n$-gon, for some integer $n \geq 2$, by identifying its edges in pairs.*

Proof. Recall that $|K|$ is obtained from the disjoint union of simplices by making some identifications. We will make these identifications in stages. At the first stage we pick, for each vertex and 1-simplex, a 2-simplex that contains it, which exists by Lemma 5.16 (3). We then identify this vertex or 1-simplex with the corresponding subset of the 2-simplex. After this process, we have a disjoint union of 2-simplices. Their edges are identified in pairs, by Lemma 5.16 (2). Note that if two edges are identified, then they lie in distinct 2-simplices. This is because the three edges of a 2-simplex in a simplicial complex are all distinct.

Pick any of these 2-simplices σ. Pick one of its edges. This is identified with an edge in another 2-simplex τ, say. Glue σ and τ together along this edge. The resulting space is homeomorphic to a square. We now pick an edge of this square which is not identified with another edge of the square. It is identified with an edge in some other 2-simplex. Attach on this 2-simplex, and so on. At each stage, we have a polygon with some of its sides identified in pairs. We stop the process when all of the sides of this polygon are identified in pairs. Let L be the subspace of K consisting of this polygon with side identifications.

We need to show that we have used up all of the 2-simplices of K. In other words, we must show that L is all of K. We note the following key fact about L: every 1-simplex of L is adjacent to precisely two 2-simplices of L. This is true both of the 1-simplices in the inside of the polygon and the 1-simplices on the boundary.

Consider any vertex of L. By construction, some 2-simplex σ_1 of L contains v. Let e_1 be any 1-simplex of σ_1 containing v. Then, by the above key fact, e_1 is incident to some other 2-simplex σ_2 of L. Let e_2 be the other 1-simplex of σ_2 containing v. This is incident to some other 2-simplex σ_3 of L, and so on. Eventually, we must return to e_1. So, the link of v in L is a simplicial circle. By the definition of a closed combinatorial surface, the link of v in K is also a simplicial circle. But the only one way that one simplicial circle can be a subcomplex of another simplicial circle is when they are equal. So, we deduce the following: for every vertex v of L, all of the simplices of K that contain v also lie in L.

Now, suppose that L is not all of K. There are two cases that we consider: either some vertex of K does not lie in L, or every vertex of K lies in L.

Suppose first that some vertex w of K does not lie in L. Since K is connected, there is an edge path joining w to some vertex of L, by Proposition 4.20. Along this path, there is some final 1-simplex that does not lie in L. After this, there is a 0-simplex v that does lie in L. But, we have already shown that every simplex that is incident to v lies in L. In particular, this is true of this 1-simplex, which is a contradiction.

Suppose now that every vertex of K lies in L. Since L is not all of K, there is some simplex not lying in L but with all of its vertices lying L. Let w be any of these vertices. Then, again, we have a vertex in L that is incident to a simplex not in L, which is a contradiction.

Thus, L is all of K. Since L is obtained from a polygon by identifying all its sides in pairs, we are done. ☐

When a closed combinatorial surface is expressed as a polygon with a complete set of side identifications, we will give some modifications to the boundary word which will not change the homeomorphism type of the surface.

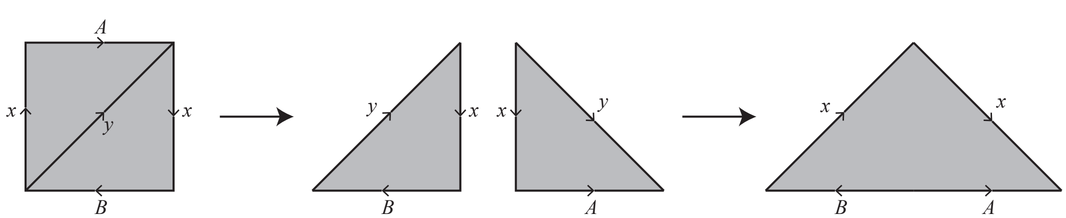
If A is a word, we use the symbol A^{-1} to denote the same collection of letters, but in reverse order, and with each letter replaced by its inverse. So, for example, if $A = xyz^{-1}$, then $A^{-1} = zy^{-1}x^{-1}$.

Lemma 5.20. *Suppose that a polygon with a complete set of side identifications has boundary word A. Then, replacing A by A^{-1}, or cyclically permuting the letters in A, does not change the resulting surface, up to homeomorphism.*

Proof. Replacing A by A^{-1} is just the same polygon, but where we read around the boundary in the opposite direction. Cyclically permuting the letters is just a change of starting point for the word. ☐

Lemma 5.21. *Suppose that a polygon with a complete set of side identifications has boundary word $xAxB$, where A and B are non-empty. Then the word $xxA^{-1}B$ gives the same surface, up to homeomorphism. Also, the word $A^{-1}xxB$ gives the same surface, up to homeomorphism.*

Proof. Let P be the polygon with boundary word $xAxB$. Let R be the arc in P running from the vertex at the end of B to the vertex at the end of A. Label it y. Cut P along this arc, forming two polygons. Glue these two polygons along the edges labelled x, forming a single polygon. (This may not be convex, but it can be easily be made so after a homeomorphism.) The boundary word of the new polygon is $yyA^{-1}B$. Now replace the y labels with x labels to get $xxA^{-1}B$. (See the figure below.)



For the second part, we use a similar decomposition of the polygon P, except we use an arc running from the start of B to the start of A.

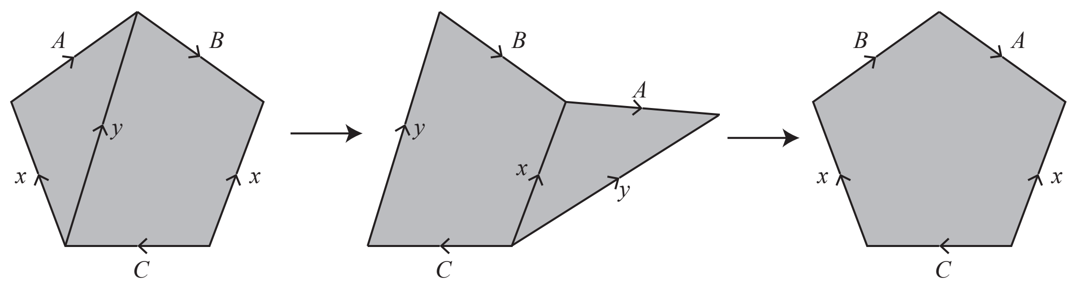
□

This has the following nice consequence.

Example 5.22. The boundary words $xy^{-1}xy$ and $xxyy$ represent homeomorphic surfaces. Hence, the Klein bottle is N_2, the surface with two crosscaps.

Lemma 5.23. *Suppose that a polygon with a complete set of side identifications has boundary word $xABx^{-1}C$. Then the word $xBAx^{-1}C$ gives the same surface up to homeomorphism.*

Proof. This is very similar to the proof of Lemma 5.21. It is summarised in the following figure. □



The above lemma states that when a word contains x and x^{-1}, we may cyclically rearrange the subword between them.

Example 5.24. The boundary words $yzy^{-1}z^{-1}xx$ and $yzyz^{-1}xx$ represent the same surface. To prove this:

$$\begin{array}{lll} yzy^{-1}z^{-1}xx & \mapsto z^{-1}xxyzy^{-1} & \text{(cyclically permute)} \\ & \mapsto z^{-1}xyxzy^{-1} & \text{(Lemma 5.23 applied to the subword between } z^{-1} \text{ and } z) \\ & \mapsto xyxzy^{-1}z^{-1} & \text{(cyclically permute)} \\ & \mapsto xxy^{-1}zy^{-1}z^{-1} & \text{(Lemma 5.21)} \\ & \mapsto y^{-1}zy^{-1}z^{-1}xx & \text{(cyclically permute)} \\ & \mapsto yzyz^{-1}xx & \text{(relabel)} \end{array}$$

Hence, adding a crosscap to a torus gives the same space as adding a crosscap to a Klein bottle.

Lemma 5.25. *If E and F are words, then $Exxyzy^{-1}z^{-1}F$ and $ExxyyzzF$ represent the same surface.*

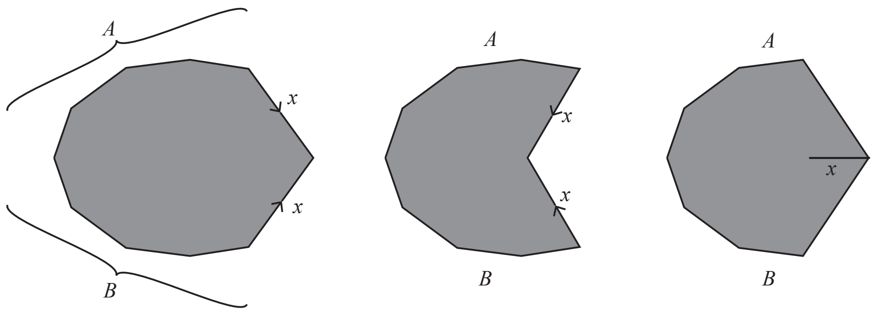
Proof. Initially, we follow the procedure given in the above example.

$$\begin{array}{lll} Exxyzy^{-1}z^{-1}F & \mapsto z^{-1}FExxyzy^{-1} & \text{(cyclically permute)} \\ & \mapsto z^{-1}xyFExzy^{-1} & \text{(Lemma 5.23)} \\ & \mapsto xyFExzy^{-1}z^{-1} & \text{(cyclically permute)} \\ & \mapsto xxE^{-1}F^{-1}y^{-1}zy^{-1}z^{-1} & \text{(Lemma 5.21)} \end{array}$$

But now we read the word in reverse order to get $Ex^{-1}x^{-1}zyz^{-1}yF$. Applying Lemma 5.21 to the subword between y and y, and relabelling, gives $ExxyyzzF$. □

Lemma 5.26. *Suppose that a polygon with a complete set of side identifications has boundary word $Axx^{-1}B$. Then the word AB gives a homeomorphic surface, provided AB has at least four letters.*

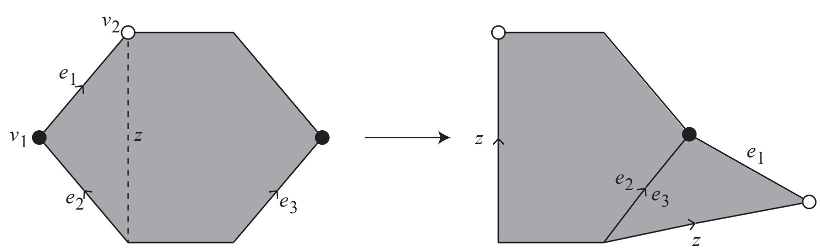
Proof. Let P be the polygon with boundary word $Axx^{-1}B$. By shifting the starting position of the word if necessary, we may assume that A and B both have at least 2 letters. Now identify the two edges labelled x and x^{-1}. The result is homeomorphic to a polygon. Its edges are identified in pairs according to the word AB. □



Let P be a polygon with a complete set of side identifications. The identifications result in vertices being identified with other vertices. In this way, the vertices of P are partitioned into equivalence classes.

Lemma 5.27. *Let P be a polygon with at least six sides and with a complete set of side identifications. Suppose that there is more than one equivalence class of vertices. Then there is another polygon P' with fewer sides than P and a complete set of side identification on P' that results in a homeomorphic surface.*

Proof. Let E be an equivalence class of vertices. We are assuming that E is not the only equivalence class of vertices. So, if one starts at a vertex in E and one runs around the boundary of P, one eventually meets a vertex not in E. So, there is some edge e_1 of the polygon with one endpoint v_1 lying in E, and the other endpoint v_2 not lying in E. In the figure below, vertices in the equivalence class E are shown in black. Let e_2 be the other edge adjacent to v_1. If this identified with e_1 and the edges point in opposite directions, then by Lemma 5.26, we can find a polygon with fewer sides and that results in a homeomorphic surface, which establishes the lemma in this case. The edges e_1 and e_2 cannot be identified with coherent orientations, because this would force v_1 and v_2 to lie in the same equivalence class, whereas we are assuming that this is not the case. Hence, e_2 is identified with some edge e_3 elsewhere in the polygon. Cut off the triangle containing the edges e_1 and e_2 from P, and attach this triangle back on to the polygon, so that e_2 and e_3 become glued. The result is a new polygon. It has the same number of edges, but the equivalence class of vertices containing v_1 has been decreased by one. However, this equivalence class is still non-empty. So, we can repeat this procedure until the equivalence class E is a single vertex. In this case, the only possibility in the above argument is that the edges e_1 and e_2 are identified with opposite orientations, and the lemma has been proved in this case. □



Proof. (of Theorem 5.18). We know, by Proposition 5.19, that any closed combinatorial surface is obtained as a polygon with a complete set of side identifications. We will prove the theorem by induction on the number of sides of this polygon.

Since a polygon must have at least three sides and since the polygons we are considering must have an even number of sides, the induction starts with the case of a square. Here, there are, up to cyclically permuting the word, relabelling the edges and possible reversing the entire word, as in Lemma 5.20, just the following possibilities:

(1) $xxyy$: This is N_2.
(2) $xxyy^{-1}$: This is N_1.
(3) $xx^{-1}yy^{-1}$: This is M_0.
(4) $xyx^{-1}y^{-1}$: This is M_1.
(5) $xyxy^{-1}$: This is homeomorphic to N_2 by Example 5.22.

So, we now assume that the polygon has at least six sides. Therefore, by Lemma 5.26, we may assume that in the boundary word, no letter is followed by its inverse. For if there is such a pair of letters, we may remove them, producing a shorter word, and the theorem then holds by induction. Similarly, by Lemma 5.27, we may assume that there is a single equivalence class of vertices.

We will now modify the boundary word of the polygon, without changing the homeomorphism type of the resulting surface. If the letter x occurs in our word as $\dots x \dots x^{-1} \dots$ (or $\dots x^{-1} \dots x \dots$), we call the occurrences of x a *reversed pair*. Similarly, $\dots x \dots x \dots$ (or $\dots x^{-1} \dots x^{-1} \dots$) is called a *standard pair*.

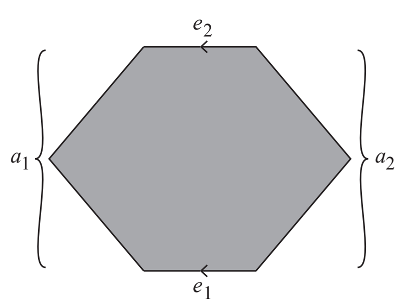
We say that two reversed pairs are *interlocking* if between x and x^{-1}, there lies precisely one of y and y^{-1}. For example, $\dots x \dots y \dots x^{-1} \dots y^{-1} \dots$ forms an interlocking pair.

Step 1. Replace the given word by AB, where $A = x_1x_1x_2x_2 \dots x_nx_n$ for some $n \geq 0$, and B contains only reversed pairs.

We can do this using Lemma 5.21, by successively moving standard pairs to the beginning of the word.

Step 2. Replace AB by AC where $C = x_1y_1x_1^{-1}y_1^{-1} \dots x_my_mx_m^{-1}y_m^{-1}$ for some $m \geq 0$.

Suppose that we have transformed the word into $Ax_1y_1x_1^{-1}y_1^{-1} \dots x_ky_kx_k^{-1}y_k^{-1}D$. (We start with the case $k = 0$ and $D = B$.) Consider the first letter x in D and its inverse x^{-1} which is also in D. We claim that they interlock with some other pair of letters y and y^{-1}. Suppose not. Denote the two edges labelled x and x^{-1} by e_1 and e_2. These two edges divide the boundary of polygon into two arcs a_1 and a_2. If x and x^{-1} did not interlock with any other pair, then each edge in the arc a_1 would be identified with another edge in a_1. Similarly, each edge in the arc a_2 would be identified with another edge in a_2. Hence, the vertices in a_1 would lie in distinct equivalence classes from the vertices in a_2. But this contradicts the assumption that there is a single equivalence class of vertices.



Since the pairs $x \dots x^{-1}$ and $y \dots y^{-1}$ interlock, the boundary word is, up to relabelling the edges, $ExFyGx^{-1}Hy^{-1}K$, where E equals $Ax_1y_1x_1^{-1}y_1^{-1} \dots x_ky_kx_k^{-1}y_k^{-1}$. Then we may apply Lemmas 5.20 and 5.21 multiple times, as follows. Each time, we cyclically permute the subword between a letter and its inverse, or we cyclically permute the entire word:

$$\begin{aligned} ExFyGx^{-1}Hy^{-1}K &\mapsto ExFyHGx^{-1}y^{-1}K && \text{(cyclically permute between } y \text{ and } y^{-1}\text{)} \\ &\mapsto ExHGFyx^{-1}y^{-1}K && \text{(cyclically permute between } x \text{ and } x^{-1}\text{)} \\ &\mapsto y^{-1}KExHGFyx^{-1} && \text{(cyclically permute entire word)} \\ &\mapsto y^{-1}HGFKExyx^{-1} && \text{(cyclically permute between } y^{-1} \text{ and } y\text{)} \\ &\mapsto Exyx^{-1}y^{-1}HGFK && \text{(cyclically permute entire word)} \end{aligned}$$

Note that the subwords E, F, G, H and K have not been reversed, and so this has not created any new standard pairs. In this way, we can transform the word into the required form.

If $A = \emptyset$, then the word is just C, and the surface is therefore M_g for some g. So, suppose that $A \neq \emptyset$.

Step 3. As $A \neq \emptyset$, then we can eliminate C by repeating the sequence of moves

$$Exxyzy^{-1}z^{-1}F \mapsto ExxyyzzF,$$

using Lemma 5.25. We end with the surface N_h, for some $h \geq 1$. □

5.6. **Distinguishing the surfaces.** We have shown that every closed combinatorial surface is homeomorphic to one of M_g or N_h, but we do not yet know that distinct surfaces in these lists are not homeomorphic. This is in fact the case:

Theorem 5.28. *None of the surfaces M_g ($g \geq 0$) or N_h ($h \geq 1$) are homeomorphic.*

The proof of this is beyond the scope of the course, and is only achieved in the 3rd year course Topology & Groups and in the 4th year course Algebraic Topology. In both of these courses, *invariants* of topological spaces are defined and studied. These are quantities that are associated with a topological space, with the property that when two spaces are homeomorphic, then their associated quantities are equal. We use the term 'quantity' somewhat vaguely here. It may be just a number, or it may be something with richer algebraic structure like a group or a ring. One of the main application of invariants is that *if* two spaces have distinct invariants, then they are not homeomorphic.

Possibly the most famous and simplest invariant of a topological space is its Euler characteristic. This is not defined for every possible topological space. But it is defined for finite simplicial complexes, as follows.

Definition 5.29. Let K be a finite simplicial complex, and let n_i be its number of i-simplices, for each integer i. Then the *Euler characteristic* of K is

$$\chi(K) = \sum_i (-1)^i n_i.$$

Note that this is a finite sum, since K has only finitely many simplices, by hypothesis.

The following is proved in the 4th year Algebraic Topology course.

Theorem 5.30. *If two finite simplicial complexes K and K' have homeomorphic topological realisations, then $\chi(K) = \chi(K')$.*

Proposition 5.31.

$$\chi(M_g) = 2 - 2g, \qquad \chi(N_h) = 2 - h.$$

Proof. This is an easy exercise, using the triangulations of M_g and N_h given in the proof of Proposition 5.17. □

This is enough to distinguish M_g and $M_{g'}$ for different values of g and g', and to distinguish N_h and $N_{h'}$ for different values of h and h'. But it leaves open the possibility that M_g and N_h are homeomorphic when $h = 2g$. This is not in fact the case, but a proof requires more sophisticated invariants.

APPENDIX A: USEFUL IDENTITIES

Let X and Y be sets, $\{U_i\}_{i\in I}$ a set of subsets of X and $\{V_j\}_{j\in J}$ a set of subsets of Y.

(1) **De Morgan laws**

$$X \setminus \bigcap_{i\in I} U_i = \bigcup_{i\in I}(X\setminus U_i)$$

$$X \setminus \bigcup_{i\in I} U_i = \bigcap_{i\in I}(X\setminus U_i)$$

(2) **Distributivity of $\bigcap$ over $\bigcup$**

$$A \bigcap \left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I}\left(A\bigcap U_i\right)$$

(3) **Images and inverse images**

Let $f: X \to Y$ be a map. Recall that for any subset A in Y its *inverse image* or *pre-image* is

$$f^{-1}(A) = \{x \in X \;;\; f(x)\in A\}\,.$$

Then

$$f(U) \subseteq V \text{ if and only if } U \subseteq f^{-1}(V)$$

$$f^{-1}(Y\setminus V) = X \setminus f^{-1}(V)$$

$$f\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f(U_i)$$

$$f\left(\bigcap_{i\in I} U_i\right) \subseteq \bigcap_{i\in I} f(U_i)$$

$$f^{-1}\left(\bigcup_{j\in J} V_j\right) = \bigcup_{j\in J} f^{-1}(V_j)$$

$$f^{-1}\left(\bigcap_{j\in J} V_j\right) = \bigcap_{j\in J} f^{-1}(V_j)$$

If $A\cap B=\emptyset$, A, B subsets of Y, then $f^{-1}(A)\cap f^{-1}(B)=\emptyset$.

Indeed assume that there exists $x \in f^{-1}(A)\cap f^{-1}(B)$. Then $f(x)\in A$ and $f(x)\in B$, which contradicts $A\cap B=\emptyset$.